CONVERGENCE

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Proposition. Every convergent sequence in \( \mathbb{R} \) is bounded.

Proof. Let \((x_n)\) be a convergent sequence in \( \mathbb{R} \), and put \( x = \lim x_n \). Choose \( k \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \), if \( n \geq k \) then \( |x_n - x| < 1 \), hence \( |x_n| < 1 + |x| \). Let
\[
M = \max\{1 + |x|, |x_1|, \ldots, |x_k|\}.
\]
Then for all \( n \in \mathbb{N} \) we have \( |x_n| \leq M \). QED

Proposition. Let \((x_n)\) and \((y_n)\) be convergent sequences in \( \mathbb{R} \). Then:

1. \( \lim(x_n + y_n) = \lim x_n + \lim y_n \).
2. \( \lim(x_n y_n) = \lim x_n \lim y_n \).
3. \( \lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} \), if \( \lim y_n \neq 0 \) and for all \( n \in \mathbb{N} \) we have \( y_n \neq 0 \).

Proof. Put \( x = \lim x_n \) and \( y = \lim y_n \).

1. Let \( \epsilon > 0 \). Choose \( k \in \mathbb{N} \) such that for all \( n \geq k \) we have
\[
|x_n - x| < \frac{\epsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\epsilon}{2}.
\]
hence
\[
|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \epsilon.
\]

2. First note that for all \( n \in \mathbb{N} \) we have
\[
|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|
= |x_n y_n - x_n y + x_n y - xy|
= |x_n(y_n - y) + (x_n - x)y|
\leq |x_n||y_n - y| + |x_n - x||y|.
\]
Since \((x_n)\) converges, it is bounded, so we can choose \( c > 0 \) such that for all \( n \in \mathbb{N} \) we have \( |x_n| \leq c \). Put \( M = \max\{c, |y|\} \). Then for all \( n \in \mathbb{N} \) we have
\[
|x_n y_n - xy| \leq M(|y_n - y| + |x_n - x|).
\]

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Now let $\epsilon > 0$. We can choose $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have
\[ |x_n - x| < \frac{\epsilon}{2M} \quad \text{and} \quad |y_n - y| < \frac{\epsilon}{2M}, \]
hence $|x_n y_n - xy| < \epsilon$.

3. By part 2, it suffices to show that
\[ \frac{1}{y_n} \to \frac{1}{y}. \]

Note that for all $n \in \mathbb{N}$ we have
\[ \frac{1}{y_n} - \frac{1}{y} = \frac{y_n - y}{y_n y}. \]

Since $y \neq 0$, we can choose $j \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq j$ then $|y_n - y| < |y|/2$, so that $|y_n| > |y|/2$, hence
\[ \left| \frac{1}{y_n} - \frac{1}{y} \right| \leq \frac{2|y - y_n|}{|y|^2}. \]

Now let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq k$ then
\[ |y_n - y| < \frac{\epsilon|y|^2}{2}. \]

Increase $k$ if necessary so that we also have $k \geq j$. Then for all $n \in \mathbb{N}$, if $n \geq k$ then
\[ \left| \frac{1}{y_n} - \frac{1}{y} \right| < \epsilon. \]

QED

**Proposition (Squeeze Theorem).** Let $(x_n), (y_n)$, and $(z_n)$ be sequences in $\mathbb{R}$. Suppose that:

- $(x_n)$ and $(z_n)$ converge to the same limit, and
- for all $n \in \mathbb{N}$ we have $x_n \leq y_n \leq z_n$.

Then $(y_n)$ converges to the common limit of $(x_n)$ and $(z_n)$.

**Proof.** Put $x = \lim x_n \ (= \lim z_n)$. Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq k$ then
\[ |x_n - x| < \epsilon \quad \text{and} \quad |z_n - x| < \epsilon, \]
hence
\[ x - \epsilon < x_n \leq y_n \leq z_n < x + \epsilon, \]
so $|y_n - x| < \epsilon$. QED

**Proposition.** Let $(x_n)$ and $(y_n)$ be convergent sequences in $\mathbb{R}$. If for all $n \in \mathbb{N}$ we have $x_n \leq y_n$, then $\lim x_n \leq \lim y_n$. 
Proof. Define a new sequence \((z_n)\) by \(z_n = y_n - x_n\), and put \(z = \lim y_n - \lim x_n\). We have \(z_n \to z\), and for all \(n \in \mathbb{N}\) we have \(z_n \geq 0\). We must show that \(z \geq 0\).

We argue by contradiction. Suppose \(z < 0\). We can choose \(k \in \mathbb{N}\) such that for all \(n \in \mathbb{N}\), if \(n \geq k\) then \(|z_n - z| < |z|\), hence \(z_n < 0\), which is a contradiction. QED