Change of Variables Theorem. If \( \phi: [a, b] \to \mathbb{R} \) is differentiable, \( \phi' \) is integrable, and \( f: \phi([a, b]) \to \mathbb{R} \) is continuous, then

\[
\int_a^b f(\phi(x))\phi'(x) \, dx = \int_{\phi(a)}^{\phi(b)} f(u) \, du.
\]

Proof. Since \( \phi \) is differentiable, it is continuous, so \( \phi([a, b]) \) is a closed bounded interval. Define \( F: \phi([a, b]) \to \mathbb{R} \) by \( F(x) = \int_{\phi(a)}^x f \). Since \( f \) is continuous, by the Fundamental Theorem of Calculus we have \( F' = f \), hence \( (F \circ \phi)' = (f \circ \phi)\phi' \) by the Chain Rule. Since \( f \) and \( \phi \) are continuous, the composition \( f \circ \phi \) is continuous, hence integrable. Since \( \phi' \) is also integrable, the product \( (f \circ \phi)\phi' \) is integrable. We apply the (other form of the) Fundamental Theorem of Calculus (twice) in the following computation to finish:

\[
\int_{\phi(a)}^{\phi(b)} f(u) \, du = F(\phi(b)) - F(\phi(a))
\]
\[
= (F \circ \phi)(b) - (F \circ \phi)(a)
\]
\[
= \int_a^b (F \circ \phi)'(x) \, dx
\]
\[
= \int_a^b f(\phi(x))\phi'(x) \, dx.
\]

QED

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