Our goal is to prove the following:

**Bolzano-Weierstrass Theorem.** Every bounded sequence of real numbers has a convergent subsequence.

We will need the following two auxiliary results:

**Proposition 1.** Every sequence in $\mathbb{R}$ has a monotone subsequence.

**Proposition 2.** Every bounded monotone sequence in $\mathbb{R}$ converges.

Let's see how these propositions are used, after which we’ll prove the propositions:

**Proof of Bolzano-Weierstrass Theorem.** Let $(x_n)$ be a bounded sequence in $\mathbb{R}$. Use Proposition 1 to find a monotone subsequence $(y_k)$. Since the original sequence $(x_n)$ is bounded, so is the subsequence $(y_k)$. Then by Proposition 2 the subsequence $(y_k)$ converges, concluding the proof. QED

**Proof of Proposition 1.** Let $(x_n)$ be a sequence in $\mathbb{R}$. We use a trick: define

$$A = \{ n \in \mathbb{N} : \text{for all } k > n \text{ we have } x_k \leq x_n \}. $$

Case 1. $A$ is finite. Choose $n_1 \in \mathbb{N}$ such that for all $n \in A$ we have $n < n_1$. For each $k = 1, 2, \ldots$ inductively choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$. Then the subsequence $(x_{n_k})$ is increasing.

Case 2. $A$ is infinite. Restrict the sequence $(x_n)$ to the infinite subset $A$ of $\mathbb{N}$ to get a subsequence $(x_{n_k})$. By construction of $A$, this subsequence is decreasing, because for all $k \in \mathbb{N}$ we have $x_{n_k} \geq x_{n_{k+1}}$. QED

**Proof of Proposition 2.** Let $(x_n)$ be a bounded monotone sequence in $\mathbb{R}$.

Case 1. $(x_n)$ is increasing. Since $(x_n)$ is bounded, we can let $x = \sup x_n$. We will show that $x_n \to x$. Let $\epsilon > 0$. Then $x - \epsilon$ is not an upper bound for $(x_n)$, so there exists $k \in \mathbb{N}$ such that
that \( x - \epsilon < x_k \). Then for all \( n \geq k \) we have:

\[
\begin{align*}
& x - \epsilon < x_k \\
& \leq x_n \quad \text{since } (x_n) \text{ is increasing} \\
& \leq x \quad \text{since } x \text{ is an upper bound for } (x_n),
\end{align*}
\]

hence \(|x_n - x| < \epsilon\).

Case 2. \((x_n)\) is decreasing. Define a sequence \((y_n)\) by \(y_n = -x_n\). Then \((y_n)\) is increasing, and is bounded since \((x_n)\) is, so from Case 1 we know that \((y_n)\) converges. Since \(x_n = -y_n\), the sequence \((x_n)\) also converges. \(\textbf{QED}\)