UNIFORM CONVERGENCE

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Exercise 1. Prove that the continuous functions \( x^n \) converge pointwise on \([0, 1]\) (as \( n \to \infty \)) to a discontinuous limit.

Exercise 2. Find an example of a subset \( A \subset \mathbb{R} \) and a sequence \((f_n)\) of functions from \( A \) to \( \mathbb{R} \) such that:

- every \( f_n \) is continuous,
- \((f_n)\) converges pointwise but not uniformly, and
- \( \lim f_n \) is continuous.

Exercise 3. For each \( n \in \mathbb{N} \) define \( f_n : [0, 1] \to \mathbb{R} \) by

\[
f_n(x) = \begin{cases} 
  n & \text{if } 0 < x < \frac{1}{n} \\
  0 & \text{if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1
\end{cases}
\]

Show that \((f_n)\) converges pointwise to an integrable function, but

\[
\int_0^1 \lim f_n \neq \lim \int_0^1 f_n.
\]

Exercise 4. Find an example of a sequence \((f_n)\) of continuous functions from \([0, 1]\) to \( \mathbb{R} \) which converges pointwise to 0 but for which \( \int_0^1 f_n \neq 0 \).

Exercise 5. Let \( g : \mathbb{R} \to \mathbb{R} \). Suppose:

- \( g \) is differentiable,
- \( g(0) = 0 \),
- \( \lim_{x \to \pm \infty} g(x) = 0 \), and
- \( g'(0) \neq 0 \).

For each \( n \in \mathbb{N} \) define \( f_n : \mathbb{R} \to \mathbb{R} \) by

\[
f_n(x) = \frac{g(nx)}{n}.
\]

Prove that \( f_n \to 0 \) uniformly, but \( f'_n(0) \neq 0 \).

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Exercise 6. Let $A \subset \mathbb{R}$, let $(f_n)$ be a uniformly convergent sequence of functions from $A \to \mathbb{R}$, let $(x_n)$ be a sequence in $A$, and let $u \in \mathbb{R}$. Put $f = \lim f_n$. Prove that if $f(x_n) \to u$, then we also have $f_n(x_n) \to u$.

Exercise 7. Let $A \subset \mathbb{R}$, and let $(f_n)$ and $(g_n)$ be uniformly convergent sequences of functions from $A$ to $\mathbb{R}$. Prove that $(f_n + g_n)$ converges uniformly.

Exercise 8. Let $A \subset \mathbb{R}$, and let $(f_n)$ and $(g_n)$ be uniformly convergent sequences of functions from $A$ to $\mathbb{R}$. Suppose that there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $x \in A$ we have $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$.

Prove that $(f_ng_n)$ converges uniformly.

Exercise 9. Show that the functions $x + 1/n$ converge uniformly on $\mathbb{R}$ (as $n \to \infty$), but $(x + 1/n)^2$ does not converge uniformly.

Exercise 10. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{x + n^2}$$

is differentiable on $[0, \infty)$, and find a formula for the derivative.