Exercise 1. Let \((x_n)\) be a convergent sequence in \(\mathbb{R}\), and put \(x = \lim x_n\). Define a new sequence \((\sigma_k)\) by
\[
\sigma_k = \frac{1}{k} \sum_{n=1}^{k} x_n.
\]
Prove that \(\sigma_k \to x\).

Hint: Break the sum into two parts, one of which has a fixed number of terms, and the other of which has terms which are close to \(x\).

Exercise 2. Let \(A \subset \mathbb{R}\) be nonempty and bounded above. Then there exists a sequence in \(A\) which converges to \(\sup A\).

Exercise 3. Let \((x_n)\) and \((y_n)\) be bounded sequences in \(\mathbb{R}\). Prove that
\[
\limsup (x_n + y_n) \leq \lim x_n + \limsup y_n.
\]

Exercise 4. Let \((x_n)\) and \((y_n)\) be sequences in \(\mathbb{R}\). Suppose \((x_n)\) converges and \((y_n)\) is bounded. Prove that
\[
\limsup (x_n + y_n) = \lim x_n + \limsup y_n.
\]

Exercise 5. Let \((x_n)\) and \((y_n)\) be sequences in \(\mathbb{R}\). Suppose \((x_n)\) converges to a positive limit, and \((y_n)\) is bounded. Prove that
\[
\limsup (x_n y_n) = \left(\lim x_n\right) \left(\limsup y_n\right).
\]

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