MAT 371 Homework 9
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1. (a) State the Mean Value Theorem for Integrals in the special case $g(x) = 1$.

(b) Let $f$ be continuous on $[a, b]$, and define $F: [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. Use the Mean Value Theorem for derivatives (that is, the original Mean Value Theorem) and the Fundamental Theorem of Calculus to give a new proof of the special case of the Mean Value Theorem for Integrals which you stated in part (a).

2. Homework 10, MAT 371 Spring 2000 (available from my web pages) states (in part): let $f: [a, b] \to [0, \infty)$ be continuous, and assume $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. The solution posted there (also available on my web pages) is elementary and argues by contradiction. Your job here is to give an alternate solution which uses the Fundamental Theorem of Calculus and properties of derivatives, applied to the function $F: [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$.

3. It is a little difficult so show directly that the improper integral
\[ \int_1^\infty \frac{\sin x}{x} \, dx \]
exists. In this problem your job is to complete the following steps in an indirect verification that this improper integral exists (and you may use all familiar properties of trig functions in this problem):

(a) For any $t > 1$, use integration by parts to replace $\int_1^t \sin x / x \, dx$ by an expression involving $\int_1^t \cos x / x^2 \, dx$.

(b) Use Remark 8.36, which is a corollary of the Comparison Theorem for Improper Integrals, to show that
\[ \int_1^\infty \frac{\cos x}{x^2} \, dx \]
exists.

(c) Use parts (a) and (b) to show that $\int_1^\infty \sin x / x \, dx$ exists.