MAT 371 HOMEWORK 10

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1. Let \( f : [a, b] \to \mathbb{R} \) be bounded.

(a) Prove that if \( f \) is integrable on \([a, b]\), then \( f \) is improperly integrable on \((a, b)\) and the improper integral coincides with the usual Riemann integral.

Hint: the hardest thing about this is believing there is anything to prove. Look carefully at our definitions of the Riemann integral and the improper integral, and you will see that there really is something to prove here. But it should be easy, using one of our elementary properties of integrals.

(b) Conversely, prove that if we only assume \( f \) is locally integrable on \((a, b)\), then \( f \) is in fact integrable on \([a, b]\).

Hint: this one is not quite so immediate, and you will probably have to use some “bare hands” technique. Remember we are still assuming \( f \) is bounded on \([a, b]\), and look carefully at the definition of locally integrable.

Note that we are only assuming \( f \) is locally integrable, not the stronger property of improper integrability, although once you solve the problem you will in fact know \( f \) is integrable, hence improperly integrable by part (a) — but you do not have to mention this.

2. Let \( f : [a, b] \to \mathbb{R} \) be bounded. Assuming the result of Problem 1 (in particular, part (b)), prove that if \( f \) has only finitely many discontinuities in \([a, b]\), then \( f \) is integrable.

Hint: assume \( f \) is continuous except possibly at \( \{t_i\}_{i=0}^n \), with \( a = t_0 < t_1 < \cdots < t_n = b \).

We included the endpoints \( a \) and \( b \) in the \( t_i \)'s for convenience — \( f \) may be continuous at \( a \) and/or \( b \), or may not be; we do not care. Show how you can use Problem 1 to prove \( f \) is integrable on each interval \([t_{i-1}, t_i]\), and then use a general property of integrals to conclude \( f \) is integrable on \([a, b]\).

Note that \( \{t_i\}_{i=0}^n \) actually forms a partition of \([a, b]\), although you are not going to use it in the usual way a partition is used.

3. (a) Let \( f : [a, b] \to \mathbb{R} \) be bounded, say \( |f(x)| \leq M \) for all \( x \in [a, b] \). Let \( P = \{x_i\}_{i=0}^n \) be a partition of \([a, b]\), let \( y \in [a, b] \setminus P \), and let \( Q = P \cup \{y\} \). Prove that

\[
|U(P) - U(Q)| \leq 2M \|P\| \quad \text{and} \quad |L(P) - L(Q)| \leq 2M \|P\|.
\]

Hint: look closely at the proof of Lemma 9.3 (1) in the Lecture Notes.

(b) The result of Part (a) can be coupled with induction to prove that \( f \) is integrable if and only if it has the following property: there exists \( I \in \mathbb{R} \) such that for all \( \epsilon > 0 \) there

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exists $\delta > 0$ such that for every partition $P = \{x_i\}_{i=0}^n$ with $\|P\| < \delta$ and any choice of $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$,

$$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - I \right| < \epsilon.$$  

In particular, if we assume $f$ is integrable, if $P_n$ denotes the partition with $n + 1$ equally spaced points, if $t_i = x_i$ for $i = 1, \ldots, n$, and if we let $\Delta x$ denote the common value of the $\Delta x_i$'s, namely $(b - a)/n$, we conclude

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(a + i \Delta x) \Delta x = \int_{a}^{b} f.$$  

You do not have to prove any of this so far. What you must prove is:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n + i} = \log 2.$$  

Hint: think of an appropriate integral $\int_{0}^{1} f$.  