Linear system

\[ m \] equations
\[ n \] unknowns \( x_1, \ldots, x_n \)
coefficients \( a_{ij}, i = 1, \ldots, m, j = 1, \ldots, n \)
constant terms \( b_1, \ldots, b_m \)

\[
\begin{align*}
  a_{11}x_1 & + a_{12}x_2 & + \cdots & + a_{1n}x_n = b_1 \\
  a_{21}x_1 & + a_{22}x_2 & + \cdots & + a_{2n}x_n = b_2 \\
  & \cdots & \cdots & \cdots \\
  a_{m1}x_1 & + a_{m2}x_2 & + \cdots & + a_{mn}x_n = b_m
\end{align*}
\]

solution: \((x_1, \ldots, x_n)\) making all equations true
solution set: set of all solutions
solve the system: find the solution set
equivalent systems:
\(\bullet\) same number of equations
\(\bullet\) same unknowns
\(\bullet\) same solution set
Example
\[ \begin{align*}
2x_1 - 3x_2 + 5x_3 - x_4 &= 7 \\
x_1 + 4x_3 - x_4 &= 7 \\
-5x_2 + x_3 &= -3
\end{align*} \]

3 equations
4 unknowns: \( x_1, x_2, x_3, x_4 \)
constant terms: 10, 2, -4
coefficients of 2nd equation: 1, 0, 4, -1
(1, 1, 2, 2) is a solution
Elimination technique
• combine equations to eliminate unknowns
• get an equivalent system at every step
• continue until solution set obvious

Matrix method
• use matrix of coefficients and constant terms rather than equations themselves
• use organized approach so final form of system and description of solution set is predictable
**Augmented matrix**

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\
\end{pmatrix}
\]

**Row operations**
- add a scalar multiple of one row to another
- multiply a row by a nonzero scalar
- swap two rows

correspond to operations on equations which produce equivalent systems

two matrices **row equivalent**: can get from one to other by row operations

augmented matrices row equivalent  \( \Rightarrow \) systems equivalent
Gauss-Jordan elimination:

Use row operations on the augmented matrix until it is in **reduced echelon form:**

- the first (from left to right) nonzero entry in each row is 1, called a **leading one**
- the leading one in each row is to the right of the leading one in the row above
- in a column containing a leading one, called a **leading column,** every other entry is 0
- any zero rows are collected at the bottom
Example

system:
\[-y = -3\]
\[-2x - y + 3z = -5\]
\[2x + 2y - 2z = 10\]

augmented matrix:
\[
\begin{pmatrix}
0 & -1 & 0 & -3 \\
-2 & -1 & 3 & -5 \\
2 & 2 & -2 & 10
\end{pmatrix}
\]

Gauss-Jordan elimination:

\[R_1 \leftrightarrow R_3:\]
\[
\begin{pmatrix}
2 & 2 & -2 & 10 \\
-2 & -1 & 3 & -5 \\
0 & -1 & 0 & -3
\end{pmatrix}
\]

\[(1/2)R_1:\]
\[
\begin{pmatrix}
1 & 1 & -1 & 5 \\
-2 & -1 & 3 & -5 \\
0 & -1 & 0 & -3
\end{pmatrix}
\]

\[2R_1 + R_2:\]
\[
\begin{pmatrix}
1 & 1 & -1 & 5 \\
0 & 1 & 1 & 5 \\
0 & -1 & 0 & -3
\end{pmatrix}
\]
more detail on last row operation:

\[
2 \begin{pmatrix} 1 & 1 & -1 & 5 \end{pmatrix} + \begin{pmatrix} -2 & -1 & 3 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -2 & 10 \end{pmatrix} + \begin{pmatrix} -2 & -1 & 3 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 5 \end{pmatrix}
\]

continue:

\[
R_2 + R_3 : \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}
\]

\[
-R_2 + R_1 : \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}
\]

\[
-R_3 + R_2 : \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}
\]

\[
2R_3 + R_1 : \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}
\]
The matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\]
is in reduced echelon form, and is the augmented matrix of the system

\[
\begin{align*}
  x &= 4 \\
  y &= 3 \\
  z &= 2 \\
\end{align*}
\]

which is solved.

Thus the original system has a unique solution: \((4, 3, 2)\)
Example
system:
\[ \begin{align*}
x + y + z &= 1 \\
x + 2y + z &= 1 \\
2x + y + 2z &= 1
\end{align*} \]
augmented matrix:
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
2 & 1 & 2 & 1
\end{pmatrix}
\]
Gauss-Jordan elimination:
\[-R_1 + R_2, -2R_1 + R_3 : \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & -1
\end{pmatrix}
\]
\[R_2 + R_3 : \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}\]

system:
\[ \begin{align*}
x + y + z &= 1 \\
y &= 0 \\
0 &= -1
\end{align*} \]
system is inconsistent: no solutions
Conclusion

If at any point in the Gauss-Jordan elimination process the augmented matrix has a row with all 0's except for a nonzero entry in the last entry, i.e., looks like

\[(0 \ 0 \ \cdots \ 0 \ \ b)\]

with \(b \neq 0\),
then the system is inconsistent
Example

system:
\[ x + y + z = 1 \]
\[ x + 2y + z = 1 \]
\[ 2x + y + 2z = 2 \]

augmented matrix:
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
2 & 1 & 2 & 2
\end{bmatrix}
\]

\[-R_1 + R_2, -2R_1 + R_3 : \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\]

\[ R_2 + R_3, -R_2 + R_1 : \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\]

system:
\[ x + z = 1 \]
\[ y = 0 \]
\[ 0 = 0 \]
Analysis

In the reduced echelon form, the leading columns are the 1st and 2nd, and these comprise the coefficients of the 1st and 2nd variables $x$ and $y$

**leading variables**: $x$, $y$

The 3rd column is not leading, and comprises the coefficients of the 3rd variable $z$

**free variable**: $z$

Solve for the leading variables in terms of the free variables:

\[
x = -z \\
y = 0
\]

solution set:

\[
\{(-z, 0, z) : z \in \mathbb{R}\}
\]

infinitely many solutions
Conclusion

If the system is **consistent** (i.e., has at least one solution), and if there is at least one free variable then the system has infinitely many solutions

Thus, for every linear system exactly one of the following is true:
- there is a unique solution
- there is no solution
- there are infinitely many solutions

\[ R = \text{reduced echelon form of augmented matrix} \]
- system consistent \iff last column of \( R \) not leading
- system has unique solution \iff last column of \( R \) not leading and very other column leading
Example

augmented matrix
\[
\begin{pmatrix}
1 & 2 & 0 & 3 & 4 & 0 \\
0 & 0 & 1 & 6 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

reduced echelon form

system inconsistent
because last column leading
(and corresponds to equation $0 = 1$)
Example

augmented matrix
\[
\begin{pmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]
reduced echelon form

1st column of augmented matrix is 0, so 1st variable \( x_1 \) is free

system
\[
\begin{align*}
x_2 &= 2 \\
x_3 &= 3
\end{align*}
\]

\( x_2, x_3 \) leading variables

solution set
\[
\{(x_1, 2, 3) : x_1 \in \mathbb{R}\}
\]
Example

augmented matrix
\[
\begin{pmatrix}
1 & 2 & 0 & 3 & 4 & 5 \\
0 & 0 & 1 & 6 & 7 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

reduced echelon form

system
\[
\begin{align*}
 x_1 + 2x_2 &+ 3x_4 + 4x_5 &= 5 \\
 x_3 + 6x_4 + 7x_5 &= 8 \\
 0 &= 0
\end{align*}
\]

leading variables: $x_1, x_3$
free variables: $x_2, x_4, x_5$

solve for leading variables in terms of free variables:
\[
\begin{align*}
x_1 &= 5 - 2x_2 - 3x_4 - 4x_5 \\
x_3 &= 8 - 6x_4 - 7x_5
\end{align*}
\]
solution set

\{ (5 - 2x_2 - 3x_4 - 4x_5, x_2, 8 - 6x_4 - 7x_5, x_4, x_5) : \\
x_2, x_4, x_5 \in \mathbb{R} \} \\

sometimes clearer to use parameters $s, t, u$ instead of the actual variables $x_2, x_4, x_5$, writing solution set in the form

\{ (5 - 2s - 3t - 4u, s, 8 - 6t - 7u, t, u) : \\
s, t, u \in \mathbb{R} \}
Matrix form of systems

system

\[ a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \]

\[ \vdots \]

\[ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \]

coefficient matrix

\[ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \]

unknown matrix

\[ \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \]

constant matrix

\[ \tilde{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \]

matrix form: \( A\tilde{x} = \tilde{b} \)

augmented matrix: \( (A \quad \tilde{b}) \)
Notation

\textbf{\textit{n}-space} \\
\mathbb{R}^n = M_{n \times 1} \\
typical element written as \(\vec{x}\) rather than \(X\), \(n\)-tuples \\
and called a \textbf{vector} or a \textbf{column matrix}

identify column matrices with ordered \(n\)-tuples \\
\[ \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \ldots, x_n) \]

Thus a solution of the system \(A\vec{x} = \vec{b}\) \\
is a vector \(\vec{x} \in \mathbb{R}^n\), \\
and the solution set is a subset of \(\mathbb{R}^n\)
Example
system \( A \vec{x} = \vec{b} \)

\[ A = \begin{pmatrix} 1 & -1 & -2 & -2 & -2 \\ 0 & 1 & 4 & 5 & -1 \\ 1 & 0 & 2 & 3 & -1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \]

augmented matrix

\[ (A \ \vec{b}) = \begin{pmatrix} 1 & -1 & -2 & -2 & -2 & -2 \\ 0 & 1 & 4 & 5 & -1 & 0 \\ 1 & 0 & 2 & 3 & -1 & 0 \end{pmatrix} \]

reduced echelon form

\[ \begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 1 \\ 0 & 1 & 4 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \]

solution

\[ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2x_3 - 3x_4 \\ 1 - 4x_3 - 5x_4 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} \]
homogeneous system
\[ A\vec{x} = \vec{0} \]

trivial solution
\[ \vec{x} = \vec{0} \]

nontrivial solution:
solution of homogeneous system
which is nontrivial

For a homogeneous system, the possibilities for the solution set are:
• only the trivial solution
  (unique solution)
• at least one nontrivial solution
  (infinitely many solutions)

In particular, a homogeneous system is automatically consistent
Example

system $A\vec{x} = \vec{0}$

$$A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 1 & 6 \end{pmatrix}$$

augmented matrix

$$(A \quad \vec{0}) = \begin{pmatrix} 3 & -2 & -2 & 0 \\ 1 & 1 & 6 & 0 \end{pmatrix}$$

last column will stay 0 throughout Gauss-Jordan elimination, so suffices to do Gauss-Jordan elimination on coefficient matrix:

reduced echelon form of $A$ is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$
system (still homogeneous)

\[
\begin{align*}
  x & + 2z = 0 \\
  y & + 4z = 0
\end{align*}
\]

leading variables \(x, y\)
free variable \(z\)

solution

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  -2z \\
  -4z \\
  z
\end{pmatrix}
\]

Observation
A homogeneous system with fewer equations than unknowns has nontrivial solutions
Example

system $A\vec{x} = \vec{0}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

reduced echelon form of $A$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only the trivial solution