Complex vector spaces — Exercises

1. Solve the system

\(-x_1 - 2x_2 = -5 - 6i\)
\((2 + i)x_1 + (3 + 2i)x_2 = 3 + 15i.\)

2. Let

\[ A = \begin{pmatrix} 0 & 1 & 2i & 2 & 11 - 2i \\ 0 & 2 & 4i & 3 & 19 - 3i \\ 0 & 2 - 2i & 4 + 4i & 4 - 4i & 18 - 26i \end{pmatrix}. \]

Find:

(a) A basis for the row space Row \(A\).

(b) A basis for the column space Col \(A\).

(c) A basis for the null space Null \(A\).

3. Does the set

\[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 - i \end{pmatrix}, \begin{pmatrix} 2 \\ 4i \\ 4 + 4i \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 - 4i \end{pmatrix}, \begin{pmatrix} 11 - 2i \\ 19 - 3i \\ 18 - 26i \end{pmatrix} \right\} \]

span \(\mathbb{C}^3\)?

4. Is the set

\[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 - i \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 - 4i \end{pmatrix}, \begin{pmatrix} 11 - 2i \\ 19 - 3i \\ 18 - 26i \end{pmatrix} \right\} \]

independent in \(\mathbb{C}^3\)?

5. Find a basis for the subspace of \(\mathbb{C}^4\) spanned by the vectors

\[ \begin{pmatrix} 1 \\ 2 \\ 3 + i \\ 1 \end{pmatrix}, \begin{pmatrix} 2 + i \\ 4 + 2i \\ 5 + 5i \\ 2 + i \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 6 + 3i \\ 2 \end{pmatrix}, \begin{pmatrix} 3 + 4i \\ 4 + 6i \\ 2 + 12i \\ 2 + 3i \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ i \\ -2 \end{pmatrix}. \]
6. In this problem, let $P_n = P_n(\mathbb{C})$ denote the complex vector space of polynomials of degree at most $n$ with complex coefficients. Define $T \in L(P_4, P_2)$ by

$$T(a + bx + cx^2 + dx^3 + ex^4) = b + 2ic + 2d + (11 - 2i)e + (2b + 4ic + 3d + (19 - 3i)e)x + ((2 - 2i)b + (4 + 4i)c + (4 - 4i)d + (18 - 26i)e)x^2.$$ 

(a) Find the matrix $[T]_{E,F}$ representing $T$ relative to the standard bases $E = \{1, x, x^2, x^3, x^4\}$ and $F = \{1, x^2\}$ of $P_4$ and $P_2$, respectively.

(b) Find a basis for the range $\text{ran} \ T$.

(c) Find a basis for the null space $\text{Null} \ T$.

(d) Find the rank of $T$.

(e) Find the nullity of $T$.

(f) Is $T$ onto?

(g) Is $T$ 1-1?

7. Define $T \in L(\mathbb{C}^2)$ by $T(\vec{x}) = A\vec{x}$, where

$A = \begin{pmatrix} 2 & i \\ -1 & 1 - i \end{pmatrix}$

and consider the two bases $E, F$ of $\mathbb{C}^2$, with

$$E = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 + i \end{pmatrix} \right\}$$

$$F = \left\{ \begin{pmatrix} 2 - i \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}.$$

(a) Directly compute $[T]_E$.

(b) Find the change of basis matrix $P$ from $F$ to $E$.

(c) Use the Change of Basis Theorem to find $[T]_F$.

8. Extend the independent set

$$\left\{ \begin{pmatrix} 2 \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -i \\ 1 \end{pmatrix} \right\}$$

to a basis of $\mathbb{C}^3$. 
9. Let \( \vec{x} = (1 + i, 2, -3i), \vec{y} = (0, 1, i) \in \mathbb{C}^3 \)

Determine:

(a) \( \langle 2\vec{x} + 3\vec{y}, -\vec{x} + (2 + i)\vec{y} \rangle \)

(b) \( \|\vec{x}\| \)

(c) whether \( \vec{x} \) and \( \vec{y} \) are orthogonal

10. Let

\[
\vec{u} = \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 1 + i \\ 0 \end{pmatrix}, \quad \text{and} \quad \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]

Apply the Gram-Schmidt Process to \( \vec{u}, \vec{v}, \vec{w} \).

11. Let

\[ E = \{(1, 1 + i), (-1 + i, 1)\}. \]

(Note that here we are using the alternate notation for vectors in \( \mathbb{C}^2 \), namely ordered pairs \( (x, y) \) rather than column matrices \( \begin{pmatrix} x \\ y \end{pmatrix} \).)

(a) Verify that \( E \) is an orthogonal basis of \( \mathbb{C}^2 \).

(b) Normalize \( E \) to obtain an orthonormal basis of \( \mathbb{C}^2 \).

12. Find the orthogonal projection of \( \vec{x} = (1, i, 0) \) onto the subspace \( W \) of \( \mathbb{C}^3 \) spanned by \( \vec{u}, \vec{v} \), where

\( \vec{u} = (2, 2, 1 + i) \) and \( \vec{v} = (0, i, -i) \).

13. Let

\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Solve the eigenvalue problem for \( A \), and diagonalize if possible.

14. In each part, determine whether the matrix is symmetric:

(a) \( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \)

(b) \( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{pmatrix} \)

(c) \( \begin{pmatrix} 1 + i & 1 + 2i \\ 1 + 2i & 3 \end{pmatrix} \)
15. In each part, determine whether the matrix is self-adjoint:

(a) \[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 3 & 2 \\
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
1 + i & 1 + 2i \\
1 + 2i & 3 \\
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
1 + i & 1 + 2i \\
1 - 2i & 3 \\
\end{pmatrix}
\]

(e) \[
\begin{pmatrix}
1 & 1 + 2i \\
1 - 2i & 3 \\
\end{pmatrix}
\]

16. Let \[
\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} -2 \\ -5 \\ -12 - 5i \\ -6i \end{pmatrix} \in \mathbb{C}^4,
\]
and let \( W = \text{span}\{\vec{x}, \vec{y}\} \). Find a basis for \( W^\perp \).

17. Suppose that \( A \) is a self-adjoint complex \( 3 \times 3 \) matrix, that the eigenvalues of \( A \) are 1 and 3, that the eigenspace associated with \( \lambda = 1 \) has a basis

\[
\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + i \\ 0 \\ 1 \end{pmatrix} \right\},
\]

and that the eigenspace associated with \( \lambda = 3 \) has a basis

\[
\left\{ \begin{pmatrix} 1 \\ -1 - i \\ 1 \end{pmatrix} \right\}.
\]

Find an orthonormal basis of \( \mathbb{C}^3 \) consisting of eigenvectors of \( A \).