Diagonalization — Exercises

1. In each part, find a basis for each eigenspace of $A$, determine whether $A$ is diagonalizable, and if it is find an invertible matrix $P$ and a diagonal matrix $D$ such that $D = P^{-1}AP$ (note that in some parts you are given the eigenvalues, while in other parts the eigenvalues are obvious):

(a) 

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(b) 

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

(c) 

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

eigenvalues: $-1, 4$

(d) 

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(e) 

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(f) 

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{pmatrix}$$

eigenvalues: $1, 2, 0$

(g) 

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}$$

eigenvalues: $1, -1$
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(h) 
\[ A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix} \]  
eigenvalues: 1, -1

(i) 
\[ A = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix} \]  
eigenvalues: 4, 0, -2

2. Let \( V \) be a 3-dimensional vector space, \( T \in L(V) \), and \( \{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \) a basis of \( V \). Assume that \( \vec{x}_1 \) and \( \vec{x}_2 \) are eigenvectors of \( T \) with associated eigenvalue 2, and \( \vec{x}_3 \) is an eigenvector with associated eigenvalue \(-5\).

(a) What is the matrix representing \( T \) relative to the basis \( \{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \)?

(b) What is the matrix representing \( T \) relative to the basis \( \{\vec{x}_1, \vec{x}_3, \vec{x}_2\} \)?

3. Define \( T \in L(P_1) \) by
\[
T(a + bx) = a + 3b + (2a + 2b)x.
\]
Find a basis of \( P_1 \) relative to which \( T \) is represented by a diagonal matrix, and find this diagonal matrix.

4. Let 
\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.
\]

(a) Find an orthonormal basis of \( \mathbb{R}^2 \) consisting of eigenvectors of \( A \). Don’t forget to normalize the vectors.

(b) Find an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( D = P^t AP \).

5. Let 
\[
A = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}.
\]

(a) Find an orthonormal basis of \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \). Hint: find a basis of each eigenspace, use the Gram-Schmidt Process if necessary to make each basis orthogonal, then normalize to make each basis orthonormal.

(b) Find an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( D = P^t AP \).