Final Exam

Time limit: 1 hour, 50 minutes. No notes, books, calculators, etc. Write your solutions on the blank pages provided — nothing on the test page(s) will be graded. Write on only one side of each page and leave reasonable margins. Print your name clearly in the upper right corner of each page, including the test page(s). Turn in your exam with the test page(s) on top. Write your final solutions clearly and in logical order — do not turn in scratch pages. You must follow all explicit instructions and show all your reasoning. This exam has 6 problems, and all problems have equal credit.

1. In this problem, only the answers will be graded — no reasons needed.

(a) True or False: If \( T : \mathbb{R}^5 \to P_4 \) is linear and onto, then \( T \) is 1-1.

(b) If \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 5 \), what is \( \det \begin{pmatrix} 2a & 2b \\ a + c & b + d \end{pmatrix} \)?

(c) If \( A \) is \( 5 \times 9 \) and \( \text{rank}(A^t) = 3 \), what is the dimension of the orthogonal complement of \( \text{Col} \ A \)?

(d) True or False: If \( A \) is \( 3 \times 4 \), then it possible for \( Ax = b \) to have a unique solution.

(e) True or False: If \( A \) and \( B \) are \( n \times n \) matrices, and \( AB \) is invertible, then \( BA \) is invertible.

(f) True or False: If \( A \) and \( B \) are \( n \times n \) matrices with the same eigenvalues, and \( A \) is singular, then so is \( B \).
2. Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \).

(a) Find the least squares solution to \( Ax = b \).

(b) Letting \( u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \) and \( v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \), use the result of part (a) to find the vector in \( \text{span}\{u, v\} \) that is closest to \( b \).

3. Suppose that \( A \) is a \( 3 \times 3 \) symmetric matrix with eigenvalues \( \lambda = 2, 4 \), and that the eigenspace associated with \( \lambda = 2 \) has basis
\[
\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}
\]
and the eigenspace associated with \( \lambda = 4 \) has basis
\[
\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.
\]
Find an orthogonal matrix \( P \) and a diagonal matrix \( D \) such that \( P^{-1}AP = D \).

4. Given that the reduced echelon form of
\[
A = \begin{pmatrix}
-6 & -1 & -15 & -7 & -71 \\
2 & 0 & 4 & 3 & 26 \\
-1 & -1 & -5 & 0 & -9 \\
1 & 0 & 2 & 1 & 10
\end{pmatrix}
\]
is
\[
R = \begin{pmatrix}
1 & 0 & 2 & 0 & 4 \\
0 & 1 & 3 & 0 & 5 \\
0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
find:

(a) a basis of the column space of \( A \);

(b) a basis of the row space of \( A \);

(c) a basis of the null space of \( A \).
5. If the reduced echelon form of \((A \ b)\) is
\[
\begin{pmatrix}
1 & 2 & 0 & 3 & 5 \\
0 & 0 & 1 & 4 & 6 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
(a) find all solutions of the system \(Ax = b\);
(b) express the 4th column \(a_4\) of \(A\) as a linear combination of the 1st and 3rd columns \(a_1, a_3\).

6. In this problem, only the answers will be graded — no reasons needed.

Let \(A = \begin{pmatrix}
3 & 1 & 2 & 4 \\
0 & -2 & 2 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 2
\end{pmatrix}\).

(a) Find the determinant of \(A\).
(b) Find the eigenvalues of \(A\).
(c) Find the rank of \(A\).
(d) Find the nullity of \(A\).
(e) Which of the standard basis vectors \(e_1, e_2, e_3, e_4\) of \(\mathbb{R}^4\) is an eigenvector of \(A\)?
(f) If \(A\) is the matrix which represents the linear operator \(T \in L(M_2)\) relative to the basis
\[E = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} ,\]
find
\[T \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} .\]