1. Find an orthonormal basis of $\mathbb{R}^4$ containing the vectors
\[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}
\]

2. (a) Find an orthogonal matrix whose first column is
\[
\begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}
\]

(b) Find an orthogonal matrix whose second row is
\[
(1/\sqrt{6} \quad 2/\sqrt{6} \quad 1/\sqrt{6})
\]

3. Let $P_2$ have the inner product
\[
\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx
\]
Let
\[
A = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}
\]
Let
\[
f(x) = a_0 + a_1 x + a_2 x^2
\]
\[
g(x) = b_0 + b_1 x + b_2 x^2
\]
be two arbitrary elements of $P_2$.
Show that
\[
\langle f, g \rangle = \begin{pmatrix} a_0 & a_1 & a_2 \end{pmatrix} A \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}
\]
4. Let
\[
E = \left\{ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}
\]
\[
F = \left\{ \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right), \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \right\}
\]
Note that \( E \) and \( F \) are both orthonormal bases of \( \mathbb{R}^2 \).

(a) Find the change of basis matrix \( P \) from \( E \) to \( F \).

(b) Verify directly that \( P \) is orthogonal.

5. Let
\[
E = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}
\]
and
\[
P = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & 2/\sqrt{6} & -1/\sqrt{6} \\
\frac{1}{\sqrt{3}} & -1/\sqrt{3} & -1/\sqrt{3}
\end{pmatrix}
\]
Note that \( E \) is an orthonormal basis of \( \mathbb{R}^3 \) and \( P \) is an orthogonal matrix.

(a) Find the unique basis \( F \) of \( \mathbb{R}^3 \) such that \( P \) is the change of basis matrix from \( E \) to \( F \).

(b) Verify directly that \( F \) is orthonormal.

6. Let \( A \in M_n \) be symmetric. Suppose
\[
\langle A \vec{x}, \vec{x} \rangle > 0 \quad \text{for all } \vec{x} \neq \vec{0}.
\]

(a) Show that every eigenvalue of \( A \) is positive.

(b) Show that \( \det A > 0 \).

(c) Show that \( \text{tr} \ A > 0 \).

7. Let \( P_2 \) have the inner product
\[
\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx.
\]
Let \( W = \text{span}\{x, x^2\} \).

(a) Find an orthonormal basis of \( W \).

(b) Extend the basis from part (a) to an orthonormal basis of \( P_2 \).

(c) Find the orthogonal projection of \( 1 + x \) onto \( W \).
8. Let

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \]

(a) Find the normal equations for the system \( A\vec{x} = \vec{b} \).

(b) Find a least squares solution of \( A\vec{x} = \vec{b} \).

(c) Find the orthogonal projection of \( \vec{b} \) onto \( \text{Col } A \).