Eigenvalues of linear operators — Exercises

1. In each part, decide whether \( \vec{u} \) is an eigenvector of \( T \), and if so determine the associated eigenvalue:

(a) \( T(\vec{x}) = A\vec{x} \) with \( A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \), \( \vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)

(b) same \( T \) as part (a), \( \vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \)

(c) same \( T \) as part (a), \( \vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)

(d) \( T(\vec{x}) = A\vec{x} \) with \( A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \), \( \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

(e) same \( T \) as part (d), \( \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)

(f) \( T \in L(M_2) \) defined by \( T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \), \( \vec{u} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \)

(g) same \( T \) as part (f), \( \vec{u} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

2. Define \( T \in L(\mathbb{R}^n) \) by

\[
T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}.
\]

Find an eigenvector with associated eigenvalue 1.

3. Let \( A \in M_n \) and \( \lambda \in \mathbb{R} \), and assume that the entries in each row of \( A \) sum to \( \lambda \). Show that \( \lambda \) is an eigenvalue of \( A \).

4. Let \( T \in L(\mathbb{R}^2) \) be rotation by \( \pi/3 \). Is \( T \) diagonalizable? Why or why not?

5. Let \( V \) be a vector space and \( T \in L(V) \), and let \( \vec{x} \) be an eigenvector of \( T \) with associated eigenvalue \( \lambda \).
(a) Show that \( \vec{x} \) is an eigenvector of \( T^2 \) with associated eigenvalue \( \lambda^2 \).

(b) Show that if \( T \) is invertible then \( \vec{x} \) is an eigenvector of \( T^{-1} \) with associated eigenvalue \( 1/\lambda \).

6. (a) Let \( V \) be the vector space of functions on \([0, 1]\) spanned by \( \{e^{2x}, x^3\} \), and define \( D \in L(V) \) by \( D(f) = f' \). Why is 2 an eigenvalue of \( D \)?

(b) Let \( V \) be the vector space of functions on \([0, 1]\) spanned by \( \{\sin x, \cos x\} \), and define \( D \in L(V) \) by \( D(f) = f' \). Show that \( D \) has no eigenvalues.