Name:

• Turn off all cell phones, pagers, and any other electronic equipment!

• No books, notes or calculators of any kind are permitted on the test.

• You may ignore any hints, but you must follow all explicit instructions.

• Do your scratch work on scratch paper, then write your solution in the space provided on the exam page. If you need more room, write on the back of the page. If you still need more room, as a last resort you may write part of your solution on scratch paper, but I prefer you not do this. Scratch paper will be provided — do not use your own scratch paper.

• Be sure to give reasons for your answers (except where explicitly told otherwise)!

• Your solutions must be complete and organized, otherwise points may be deducted.

• There are 5 problems, and all problems have equal credit.
1. Let $V$ be a vector space, and let $x_1, \ldots, x_n \in V$. In each part, complete the indicated definition:

(a) \(\{x_1, \ldots, x_n\}\) is independent if \dots

(b) \(\{x_1, \ldots, x_n\}\) spans $V$ if \dots

(c) \(\{x_1, \ldots, x_n\}\) is a basis of $V$ if \dots
2. In this problem, only the answer will be graded; you do **not** need to show any work!

(a) Let $W$ be a subspace of $\mathbb{R}^3$. What are all the possible values of $\dim W$?

Solution: $0, 1, 2, 3$

(b) Let $E$ be a basis of $\mathbb{R}^9$. How many vectors does $E$ have?

Solution: $9$

(c) Let $V$ be a vector space, and let $x_1, x_2, x_3, x_4 \in V$. Assume that $\{x_1, x_2, x_3\}$ is independent and $x_4 \in \text{span}\{x_1, x_2, x_3\}$. What is the dimension of the subspace spanned by $\{x_1, x_2, x_3, x_4\}$?

Solution: $3$

(d) Are the vectors \[
\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}\] in $\mathbb{R}^4$ independent?

Solution: no

(e) Do the vectors \[
\begin{pmatrix} 4 \\ 4 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}\] span $\mathbb{R}^4$?

Solution: no

(f) Are the vectors \[
\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}\] in $\mathbb{R}^3$ independent?

Solution: no
3. Prove that the set

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$$

is a subspace of $M_2$. You must use only the definition of subspace.
4. In each part of this problem, either give an example with the indicated properties or state that it is impossible. You do not need to give reasons.

(a) a dependent set which spans $\mathbb{R}^2$

Solution: \[
\{(1,0), (0,1), (1,1)\}
\]

(b) a basis of $M_{2\times3}$

Solution: \[
\{(1,0,0), (0,1,0), (0,0,1), (0,0,0), (0,0,0), (0,0,0)\}
\]

(c) an independent subset $E$ of the vector space of polynomials of degree at most 2 such that $E$ has 4 elements

impossible
5. Let $V$ be a vector space, and let $x, y, z \in V$. Assume that $z \neq 0$, and that $z$ is a linear combination of $x$ and $y$. Prove that at least one of the following is true:

- $x \in \text{span}\{y, z\}$, or
- $y \in \text{span}\{x, z\}$.

Hint: in an expression of $z$ as a linear combination of $x$ and $y$, consider nonzero coefficients.