Let $V$ and $W$ be sets. Officially, a function from $V$ to $W$ would be defined as a subset of the Cartesian product $V \times W$ satisfying certain properties. However, informally we regard a function from $V$ to $W$ as a rule which associates to each element of $V$ a unique element of $W$. We write $f: V \to W$ to mean $f$ is a function from $V$ to $W$.

The element of $W$ that a function $f: V \to W$ associates to an element $x \in V$ is the image of $x$ under $f$, or the value of $f$ at $x$, and denoted $f(x)$.

The main thing is that two functions $f, g: V \to W$ are equal precisely when they have the same values, i.e., $f = g$ if and only if for all $x \in V$ we have $f(x) = g(x)$.

Let $f: V \to W$.

1. $V$ is the domain of $f$.
2. If $Z \subset V$, the image of $Z$ under $f$ is the following subset of $W$:
   \[ f(Z) := \{ f(x) : x \in Z \} = \{ y \in W : \text{there exists } x \in Z \text{ such that } y = f(x) \} \]
3. If $Z \subset W$, the inverse image of $Z$ under $f$ is the following subset of $V$:
   \[ f^{-1}(Z) := \{ x \in V : f(x) \in Z \} \]
4. The range of $f$ is
   \[ \text{ran } f := f(V) \]
5. $f$ is one-to-one, written 1-1, if for all $x, y \in V$, if $f(x) = f(y)$ then $x = y$.
6. $f$ is onto if $\text{ran } f = W$, i.e., if for all $y \in W$ there exists $x \in V$ such that $y = f(x)$.
7. $f$ is a 1-1 correspondence if it is 1-1 onto (i.e., both 1-1 and onto).

Note that the domain of $f: V \to W$ is $f^{-1}(W)$. Also note that equal functions have the same domain. But more is true: if $f: V \to W$, then in this course we make the convention that both sets $V$ and $W$ be part of the function $f$. The set $W$ is called the codomain of $f$, but we won’t need this word. To emphasize: if $f: V \to W$, $g: V \to Z$, and for all $x \in V$ we have $f(x) = g(x)$, but $W \neq Z$, then $f \neq g$.

The identity function on a set $V$ is the function $I_V: V \to V$ defined by
\[ I_V(x) = x. \]

Let $V, W$, and $Z$ be sets, and let $f: V \to W$ and $g: W \to Z$. The composition of $f$ and $g$ is the function
\[ g \circ f : V \to Z \]
defined by
\[ (g \circ f)(x) = g(f(x)). \]
This can be visualized by the following “commutative diagram”:

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
& \searrow^{g \circ f} & \downarrow^{g} \\
& & Z.
\end{array}
\]

In this setting, we have:

1. If \( f \) and \( g \) are both 1-1, then so is \( g \circ f \);
2. If \( f \) and \( g \) are both onto, then so is \( g \circ f \);
3. If \( g \circ f \) is 1-1, then so is \( f \);
4. If \( g \circ f \) is onto, then so is \( g \).

If \( f : V \to W \) is invertible if it is 1-1 onto, in which case there is a unique inverse function \( f^{-1} : W \to V \) satisfying

\[
f^{-1} \circ f = I_V \quad \text{and} \quad f \circ f^{-1} = I_W.
\]

\( f^{-1} \) can be defined as follows: for each \( y \in W \), \( f^{-1}(y) \) is the unique \( x \in V \) such that \( f(x) = y \). Thus, for all \( x \in V \) and \( y \in W \) we have

\[
f(x) = y \quad \text{if and only if} \quad x = f^{-1}(y).
\]

This also characterizes the inverse function, i.e., there is no other function which is related to \( f \) in this way.

If \( f \) is an invertible function, then

\[
(f^{-1})^{-1} = f.
\]

If \( f \) and \( g \) are invertible composable functions, then

\[
(g \circ f)^{-1} = f^{-1} \circ g^{-1}.
\]