Be sure to review the “Preliminaries on numbers”.

In this section we switch from real to complex scalars. There is almost no change in the vector space structure, but the effect on the inner product spaces is nontrivial, albeit straightforward. Our interest lies in the following: when we used real scalars, it was possible for linear operators to have no eigenvalues; this defect does not occur with complex scalars. In particular, this allows us to finally see why every symmetric linear operator on a finite-dimensional inner product space has an eigenvalue. Interestingly, even for real scalars the detour through complex inner product spaces is the easiest way to prove this result about symmetric operators.

To emphasize the distinction between the vector spaces we have studied before now and the ones we’ll study here, we call the earlier ones real vector spaces, to indicate that we use real numbers as scalars.

**Complex vector spaces.** A complex vector space is a set $V$ equipped with two operations called addition and scalar multiplication, which behave just as for real vector spaces, except that the scalars are taken from the complex numbers $\mathbb{C}$ rather than the real numbers $\mathbb{R}$.

**Example.** For each $n \in \mathbb{N}$ define

$$\mathbb{C}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{C} \right\}.$$ 

The operations of addition and scalar multiplication are defined just as for $\mathbb{R}^n$, except that the coordinates $x_1, \ldots, x_n$ are complex numbers rather than real numbers.

Essentially everything we said about real vector spaces carries over verbatim for complex vector spaces — throughout we just substitute $\mathbb{C}$ for $\mathbb{R}$. In particular, $m \times n$ matrices with complex entries have the same properties as for real entries, and we use the same notation $M_{m \times n}(\mathbb{R})$, but if we want to avoid confusion we can distinguish the two possibilities by $M_{m \times n}(\mathbb{R})$ and $M_{m \times n}(\mathbb{C})$, and similarly for square matrices $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ and invertible matrices $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$.

Again, we emphasize that all results for vector spaces remain true for complex scalars.
**Complex inner product spaces.** A *complex inner product space* is a complex vector space $V$ equipped with another operation, called the *inner product*, and denoted the same as for real inner product spaces, and having almost all the same properties, except: for all $x, y \in V$ we have

$$\langle x, y \rangle = \overline{\langle y, x \rangle}.$$ 

Consequently, the inner product is linear in the 1st variable:

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \text{and} \quad \langle cx, y \rangle = c\langle x, y \rangle,$$

but *conjugate-linear* in the 2nd variable:

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \text{and} \quad \langle x, cy \rangle = \overline{c}\langle x, y \rangle.$$

We still have

$$\langle x, x \rangle > 0 \quad \text{if} \ x \neq 0,$$

and we define the *norm* as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$ 

All other basic results for inner product spaces carry over, e.g., the *Cauchy-Schwarz Inequality*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

(although we need a slightly different proof with complex scalars) and the *Triangle Inequality*

$$\|x + y\| \leq \|x\| + \|y\|.$$

**Example.** The *dot product* of $x, y \in \mathbb{C}^n$ is

$$x \cdot y := \sum_{i=1}^{n} x_i \overline{y_i}.$$ 

We conjugate the coordinates of the 2nd variable $y$ to get a complex inner product. In this example the Cauchy-Schwarz Inequality implies that for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{C}$ we have

$$\left| \sum_{i=1}^{n} x_i \overline{y_i} \right|^2 \leq \left( \sum_{i=1}^{n} |x_i|^2 \right) \left( \sum_{i=1}^{n} |y_i|^2 \right).$$

Note that, unlike the case of $\mathbb{R}^n$, we cannot just write, e.g., $x_i^2$, because for complex numbers this would not generally give a nonnegative number; we must use the square of the absolute value $|x_i|$, which does always give a nonnegative number. Similarly, we must use the absolute value of the sum on the left-hand side.

Orthogonality and orthonormality are the same as for real inner product spaces. We still have the *Pythagorean Theorem*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{if} \ x \perp y,$$
the expansion
\[ x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i \]
if \( \{u_1, \ldots, u_n\} \) is an orthonormal basis of \( V \), and the formula
\[ A_{ij} = \langle T(x_j), y_i \rangle \]
for the entries of the matrix \( A \) representing \( T \in L(V,W) \) relative to orthonormal bases \( \{x_1, \ldots, x_n\} \) of \( V \) and \( \{y_1, \ldots, y_m\} \) of \( W \).

We also have orthogonal projections and the Gram-Schmidt Process. Again, we repeat that all the basic results for real inner product spaces carry over with minor changes to complex inner product spaces.

**Adjoints.** However, the conjugate-linearity of the inner product in the second variable results in significant changes when we try to “complexify” the transpose — in particular, we won’t use the term “transpose” anymore. Let \( V \) be a finite-dimensional complex inner product space. It is still true that for every \( f \in L(V, \mathbb{C}) \) there exists a unique \( y \in V \) such that
\[ f(x) = \langle x, y \rangle \quad \text{for} \quad x \in V. \]
However, the assignment
\[ L(V, \mathbb{C}) \xrightarrow{f \mapsto y} V \]
is no longer linear (in fact it is conjugate-linear, although we will not use this). Now let \( W \) be another finite-dimensional complex inner product space, and let \( T \in L(V,W) \). Then for \( y \in W \) we can still define \( f \in L(V, \mathbb{C}) \) by
\[ f(x) = \langle T(x), y \rangle, \]
but now we denote by \( T^*(y) \) the unique vector in \( V \) such that
\[ \langle x, T^*(y) \rangle = f(x) \quad \text{for} \quad x \in V, \]
and we have
\[ \langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for} \quad x \in V, y \in W. \]
\( T^*: W \to V \) is still linear (the two conjugates “cancel out”), and it is called the *adjoint* of \( T \). The assignment
\[ L(V,W) \xrightarrow{T \mapsto T^*} L(W,V) \]
is still 1-1 onto, but it’s no longer linear, rather it is conjugate-linear: we do have
\[ (T + S)^* = T^* + S^*, \]
but for \( c \in \mathbb{C} \) we have
\[ (cT)^* = \overline{c}T^*. \]
All other properties of adjoint faithfully parallel those of the transpose for real inner product spaces.

If $A$ represents $T$ relative to orthonormal bases $E$ of $V$ and $F$ of $W$, then the matrix representing $T^*$ relative to $F$ and $E$ is called the adjoint of $A$, denoted $A^*$, and is actually the conjugate transpose:

$$(A^*)_{ij} = \overline{A_{ji}}.$$

**Example.** If

$$A = \begin{pmatrix} 1 + 2i & 3 + 4i & 5 + 6i \\ 7 + 8i & 9 + 10i & 11 + 12i \end{pmatrix},$$

then

$$A^* = \begin{pmatrix} 1 - 2i & 7 - 8i \\ 3 - 4i & 9 - 10i \\ 5 - 6i & 11 - 12i \end{pmatrix}.$$

**Example.** For all $x, y \in \mathbb{C}^n$ we have

$$x \cdot y = y^* x.$$

Note that we could define transpose for complex matrices, but we will have no use for this.

$T \in L(V)$ is self-adjoint if $T = T^*$, and similarly for matrices.

**Example.** The matrix

$$\begin{pmatrix} 3 & 4 + 5i \\ 4 - 5i & 7 \end{pmatrix}$$

is self-adjoint.

Again, there is a little change when we try to complexify orthogonal operators — we use adjoints rather than transposes, and the term is unitary rather than orthogonal: more precisely, if $V$ is a finite-dimensional complex inner product space, then $T \in L(V)$ is unitary if $T^* = T^{-1}$, and similarly for matrices in $M_n(\mathbb{C})$.

**Example.** For $\theta_1, \ldots, \theta_n \in \mathbb{R}$, the diagonal matrix

$$\begin{pmatrix} e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\theta_n} \end{pmatrix}$$

is unitary, because

$$e^{i\theta} = e^{-i\theta} \quad \text{for } i = 1, \ldots, n.$$
The earlier results on orthogonal operators carry over appropriately, \textit{e.g.}, a linear operator on a finite-dimensional complex inner product space is unitary if and only if it preserves orthonormality, and a matrix in $M_n(C)$ is unitary if and only if its columns form an orthonormal basis of $C^n$, if and only if its adjoint is unitary.

A linear operator on a finite-dimensional complex inner product space is \textit{unitarily diagonalizable} if there is an orthonormal basis relative to which the operator is represented by a diagonal matrix. and a matrix in $M_n(C)$ is \textit{unitarily diagonalizable} if there is a unitary matrix which diagonalizes it.

\textbf{Eigenvalues and determinants.} Determinants are defined the same way for complex matrices as for real ones, and have the same properties. Most importantly, a complex square matrix is invertible if and only if its determinant is nonzero. Consequently, we retain the use of the characteristic polynomial: if $A \in M_n(C)$ and $\lambda \in C$, then $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I) = 0$. However, here is a crucial difference: by the Fundamental Theorem of Algebra, every polynomial with complex coefficients has a root in the complex numbers. Thus:

\textbf{Theorem.} Every element of $M_n(C)$ has an eigenvalue in $C$.

\textbf{Corollary.} Let $V$ be a finite-dimensional complex vector space and $T \in L(v)$. Then $T$ has an eigenvalue.

Consequently, to adequately study eigenvalues it’s imperative to allow complex scalars.

\textbf{Self-adjointness and eigenvalues.} Now we come to the hidden payoff for allowing complex scalars: we can complete the proof of the Spectral Theorem.

\textbf{Theorem.} Let $V$ be a finite-dimensional complex inner product space, and let $T \in L(V)$ be self-adjoint. Then every eigenvalue of $T$ is real.

\textbf{Proof.} Let $\lambda$ be an eigenvalue of $T$, and choose an associated eigenvector $x$. Then:

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle T(x), x \rangle = \langle x, T(x) \rangle = \langle x, \lambda x \rangle = \overline{\lambda} \langle x, x \rangle.$$ 

Since $x$ is an eigenvector, we have $x \neq 0$, so $\langle x, x \rangle \neq 0$, hence $\lambda = \overline{\lambda}$. Therefore $\lambda \in \mathbb{R}$. \textbf{QED}

\textbf{Corollary.} Let $A \in M_n(C)$ be self-adjoint. Then every eigenvalue of $A$ is real.

\textbf{Corollary.} Let $A \in M_n(\mathbb{R})$ be symmetric, Then $A$ has an eigenvalue in $\mathbb{R}$.

\textbf{Proof.} Since $A$ is symmetric and has real entries, it may be regarded as a self-adjoint element of $M_n(\mathbb{C})$. Thus it has an eigenvalue in $\mathbb{C}$. But since it is self-adjoint, every eigenvalue is real. Thus $A$ has an eigenvalue in $\mathbb{R}$. \textbf{QED}
Note that in the above proof, once we found that $A$ had a real eigenvalue $\lambda$, by definition this means there is a nonzero $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. To regard $\lambda$ as an eigenvalue of $A$ in $M_n(\mathbb{R})$, we need to know that there exists a nonzero $x \in \mathbb{R}^n$ such that $Ax = \lambda x$. But this is not an issue, because we know that $\lambda$ is a root of the characteristic equation, so the existence of associated eigenvectors in $\mathbb{R}^n$ is automatically guaranteed.

We are finally ready to fill in the one remaining gap in the Spectral Theorem for Symmetric Operators:

**Corollary.** let $V$ be a finite-dimensional real inner product space, and let $T \in L(V)$ be symmetric. Then $T$ has an eigenvalue.

**Corollary.** Every symmetric matrix with real entries has an eigenvalue.

There is much more to say, but we’ll stop here.