Throughout this section, $A$ will be an $n \times n$ matrix.

We will not prove all the properties of determinants, but we will show how some of these properties follow from others. Roughly speaking, the properties can be proved using induction on the size of the matrix. All the properties are trivial for $1 \times 1$ matrices and obvious for $2 \times 2$ matrices.

**Theorem.** $\det(A) = \det(A^t)$.

**Example.** Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

Then

$$\det(A) = \det(\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}) - 2 \det(\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}) + 3 \det(\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix})$$

$$= 45 - 48 - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0,$$=

and

$$\det(A^t) = \det(\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix})$$

$$= \det(\begin{pmatrix} 5 & 8 \\ 6 & 9 \end{pmatrix}) - 4 \det(\begin{pmatrix} 2 & 8 \\ 3 & 9 \end{pmatrix}) + 7 \det(\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix})$$

$$= 45 - 48 - 4(18 - 24) + 7(12 - 15)$$

$$= -3 + 24 - 21 = 0.$$

**Corollary.** The determinant of a triangular matrix equals the product of the diagonal matrices.

**Proof.** Let $A \in M_n$ be triangular. First, if $A = (a_{ij})$ is lower triangular, then $\det(A) = a_{11} \det(A_{11})$, and $A_{11} \in M_{n-1}$ is lower triangular, so $\det(A) = a_{11} \cdots a_{nn}$ by induction on $n$ (because it’s trivially true if $n = 1$).

On the other hand, if $A$ is upper triangular, then $A^t$ is lower triangular with the same diagonal entries as $A$, so because $\det(A^t) = \det(A)$ we are done. QED
Example. We have
\[
\begin{vmatrix}
2 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 \\
5 & 5 & 5 & 5
\end{vmatrix} = 2 \begin{vmatrix}
3 & 0 & 0 \\
4 & 4 & 0 \\
5 & 5 & 5
\end{vmatrix}
\]
\[
= 2(3) \begin{vmatrix}
4 & 0 \\
5 & 5
\end{vmatrix}
\]
\[
= 2(3)(4)(5).
\]

Corollary. \(\det(I) = 1\).

It turns out that many of the important properties of the determinant are expressed in terms of the rows of \(A\). These are elements of the vector space \(M_{1\times n}\). The columns are elements of the vector space \(M_{n\times 1} = \mathbb{R}^n\), which we understand very well. We can work with \(M_{1\times n}\) just as easily.

It is obvious from the definition that, if rows 2, \ldots, \(n\) of \(A\) are fixed, then \(\det(A)\) is a linear function of the 1st row.

Example. Let \(n = 2\), and fix the 2nd row of \(A\) to be \((3 \ -2)\). Then as a function of the first row \((x \ y)\) of \(A\), the determinant is
\[
\det(A) = \det\begin{pmatrix} x & y \\ 3 & -2 \end{pmatrix} = -2x - 3y,
\]
which is obviously linear.

In preparation for our next property, we introduce a little more terminology: for each \(i, j = 1, \ldots, n\), \((-1)^{i+j}\det(\tilde{A}_{ij})\) is the \(ij\)-cofactor of \(A\), or (somewhat ambiguously) the cofactor of the \(ij\)-entry of \(A\). The definition of \(\det(A)\) is the sum of the products of the entries of the 1st row times their cofactors, and this is called cofactor expansion along the 1st row.

Example. Let
\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 7 & 8 & 9 \end{pmatrix}.
\]

The 11-cofactor, or the cofactor of the 11-entry, is
\[
(-1)^{1+1} \det\begin{pmatrix} 1 & 1 \\ 8 & 9 \end{pmatrix} = 9 - 8 = 1,
\]
and the 23-cofactor, or the cofactor of the 23-entry, is
\[
(-1)^{2+3} \det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = -(8 - 14) = 6.
\]
Note that the 11- and the 23-entries of $A$ are both equal to 1 (as are two other entries!), so it would be ambiguous to refer to the “cofactor of the entry 1”.

**Theorem.** The determinant of $A$ can be computed by expansion along any row or any column, i.e., if $A = (a_{ij})$, then:

1. for each $i = 1, \ldots, n$ we have
   \[
   \det(A) = \sum_{j=1}^{n} (-1)^{i+j}a_{ij} \det(\tilde{A}_{ij});
   \]

2. for each $j = 1, \ldots, n$ we have
   \[
   \det(A) = \sum_{i=1}^{n} (-1)^{i+j}a_{ij} \det(\tilde{A}_{ij});
   \]

We won’t prove the first part; roughly speaking, the second part follows by taking the transpose.

**Example.** We’ve already computed the determinant of the matrix
\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\]
two ways: by definition, i.e., cofactor expansion along the first row, and by applying the definition to $\det(A^t)$. Let’s see how the second method is essentially the same as cofactor expansion along the 1st column:

\[
\begin{align*}
\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 7 \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \\
&= \det \begin{pmatrix} 5 & 8 \\ 6 & 9 \end{pmatrix} - 4 \det \begin{pmatrix} 2 & 8 \\ 3 & 9 \end{pmatrix} + 7 \det \begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} \\
&= \text{(transposing the $2 \times 2$ matrices)},
\end{align*}
\]

which is exactly the computation of the determinant of $A^t$ by cofactor expansion along the first row.

**Example.** Let
\[
A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & -1 \end{pmatrix}
\]
Let’s compute \( \det(A) \) first by cofactor expansion along the 2nd row:

\[
\begin{vmatrix}
1 & 3 & 1 \\
2 & 0 & 1 \\
2 & 2 & -1
\end{vmatrix} = -2 \begin{vmatrix}
3 & 1 \\
2 & -1
\end{vmatrix} + 0 \begin{vmatrix}
1 & 1 \\
2 & -1
\end{vmatrix} - \begin{vmatrix}
1 & 3 \\
2 & 2
\end{vmatrix}
\]

\[= -2(-5) - (2 - 6) = 14.\]

Next let’s compute \( \det(A) \) by cofactor expansion along the 3rd column:

\[
\begin{vmatrix}
1 & 3 & 1 \\
2 & 0 & 1 \\
2 & 2 & -1
\end{vmatrix} = \begin{vmatrix}
2 & 0 \\
2 & 2
\end{vmatrix} - \begin{vmatrix}
1 & 3 \\
2 & 2
\end{vmatrix} - \begin{vmatrix}
1 & 3 \\
2 & 0
\end{vmatrix}
\]

\[= 4 - (4) - (6) = 14.\]

Corollary.

1. For each \( i = 1, \ldots, n \), if all the rows except the \( i \)th row of \( A \) are fixed, then \( \det(A) \) is a linear function of the \( i \)th row.

2. For each \( j = 1, \ldots, n \), if all the columns except the \( j \)th column of \( A \) are fixed, then \( \det(A) \) is a linear function of the \( j \)th column.

Proof. 1. This follows immediately from cofactor expansion along the \( i \)th row, once we notice that the cofactors of the \( i \)th row do not involve the elements of the \( i \)th row.

2. This is similar, using cofactor expansion along the \( j \)th column. \( \Box \)

Symbolically, part 1 of the above corollary says

\[
\begin{vmatrix}
| a_1 |
| \vdots |
| a_i' + a_i'' |
| \vdots |
| a_n |
\end{vmatrix} = \begin{vmatrix}
| a_1 |
| \vdots |
| a_i' |
| \vdots |
| a_n |
\end{vmatrix} + \begin{vmatrix}
| a_1 |
| \vdots |
| a_i'' |
| \vdots |
| a_n |
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
| a_1 |
| \vdots |
| ca_i |
| \vdots |
| a_n |
\end{vmatrix} = c \begin{vmatrix}
| a_1 |
| \vdots |
| a_i |
| \vdots |
| a_n |
\end{vmatrix} \quad \text{for } c \in \mathbb{R},
\]

where the \( a_k, a'_i, \) and \( a''_i \) are in \( M_{1 \times n} \). Part 2 (for columns) can be similarly symbolically written.
Example.

$$\det \begin{pmatrix} a & b & c \\ d' + d'' & e' + e'' & f' + f'' \\ g & h & i \end{pmatrix} = \det \begin{pmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{pmatrix} + \det \begin{pmatrix} a & b & c \\ d'' & e'' & f'' \\ g & h & i \end{pmatrix}.$$ 

Example.

$$\det \begin{pmatrix} a & -3b & c \\ d & -3e & f \\ g & -3h & i \end{pmatrix} = -3 \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$ 

Example.

$$\det \begin{pmatrix} 2a & 2b & 6c \\ d & e & 3f \\ g & h & 3i \end{pmatrix} = 6 \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$ 

Example. Fixing the 1st and 3rd rows of a $3 \times 3$ matrix $A$ to be $(2 \ -4 \ 3)$ and $(1 \ 7 \ 5)$, respectively, as a function of the 2nd row $(x \ y \ z)$ of $A$ we have

$$\det(A) = \det \begin{pmatrix} 2 & -4 & 3 \\ x & y & z \\ 1 & 7 & 5 \end{pmatrix} = -x \det \begin{pmatrix} -4 & 3 \\ 7 & 5 \end{pmatrix} + y \det \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} - z \det \begin{pmatrix} 2 & -4 \\ 1 & 7 \end{pmatrix} = 41x + 7y - 18z.$$ 

Example. Fixing the 1st and 2nd columns of a $3 \times 3$ matrix $A$ to be $\left(\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{3} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -\frac{4}{5} \\ \frac{1}{6} \end{smallmatrix}\right)$, as a function of the 3rd column $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ of $A$ we have

$$\det(A) = \det \begin{pmatrix} 1 & -4 & x \\ 2 & 5 & y \\ 3 & 6 & z \end{pmatrix} = x \det \begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} - y \det \begin{pmatrix} 1 & -4 \\ 3 & 6 \end{pmatrix} + z \det \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix} = -3x - 18y + 13z.$$ 

Corollary. If $A$ has a zero row or column, then $\det(A) = 0$. 

Example. We have

$$\det \begin{pmatrix} 1 & 2 & 3 & 0 & 4 \\ 5 & 6 & 7 & 0 & 8 \\ 9 & 10 & 11 & 0 & 12 \\ 13 & 14 & 15 & 0 & 16 \\ 17 & 18 & 19 & 0 & 20 \end{pmatrix} = 0$$
because the 4th column is 0.

**Corollary.** If $A$ is in reduced row echelon form, then $\det(A) \neq 0$ if and only if $A = I$.

**Proof.** If $A = I$, then $\det(A) = 1 \neq 0$. On the other hand, if $A \neq I$, then, because $A$ is a square matrix in reduced row echelon form, the last row of $A$ is 0, so $\det(A) = 0$. \hspace{1cm} \text{QED}

The following is another of those results we will not prove. Again, it is trivial (actually, vacuous) for $1 \times 1$ matrices, and obvious for $2 \times 2$ matrices, and it can be proved by induction on $n$ if $A \in M_n$.

**Theorem.** If the matrix $B$ is obtained from $A$ by switching two rows or two columns, then

$$\det(B) = -\det(A).$$

**Example.**

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2,$$

and

$$\det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = (3)(2) - (4)(1) = 2.$$

**Example.**

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

(switching the 2nd and 4th columns)

$$= -(1)(2)(3)(4)$$

(because it’s upper triangular)

$$= -24.$$

**Corollary.** If $A$ has two equal rows or two equal columns, then $\det(A) = 0$.

**Proof.** If $A$ has two equal rows, then $A$ is unchanged when these two rows are switched, but $\det(A)$ is multiplied by $-1$. The only real number which is equal to its negative is 0. Similarly for columns. \hspace{1cm} \text{QED}

The following theorem shows how the properties we have recorded above allow us to keep track of the effect of elementary row operations on the determinant.

**Theorem.** 1. If $B$ is obtained from $A$ by an elementary row operation of type 1, \textit{i.e.}, by multiplying a row by a nonzero scalar $c$, then

$$\det(B) = c \det(A).$$
2. If $B$ is obtained from $A$ by an elementary row operation of type 2, i.e., adding a scalar multiple of one row to a different row, then

$$\det(B) = \det(A).$$

3. If $B$ is obtained from $A$ by an elementary row operation of type 3, i.e., switching two rows, then

$$\det(B) = -\det(A).$$

Proof. We already know properties 1 and 3, so it remains to prove property 2: if $B$ is obtained from $A$ by adding $c$ times row $i$ to row $j$ (where without loss of generality $i < j$), then

$$\det(B) = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j + ca_i \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} + c \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ a_j \\ \vdots \\ a_n \end{pmatrix} = \det(A)$$

(because the 1st matrix is $A$ and the 2nd matrix has row $i = \text{row } j$).

QED

Corollary.

1. If $E$ is an elementary matrix of type 1, i.e., $E$ is obtained from the identity matrix by multiplying some row by a nonzero scalar $c$, then $\det(E) = c$.

2. If $E$ is an elementary matrix of type 2, i.e., $E$ is obtained from the identity matrix by adding a scalar multiple of one row to a different row, then $\det(E) = 1$.

3. If $E$ is an elementary matrix of type 3, i.e., $E$ is obtained from the identity matrix by switching two rows, then $\det(E) = -1$.

Now we are ready to show how the above properties of determinants imply the most important one:
**Theorem.** A is invertible if and only if \( \det(A) \neq 0 \).

**Proof.** Let \( U \) be the reduced row echelon form of \( A \). Then \( A \) is invertible if and only if \( U = I \). \( U \) is obtained from \( A \) by a finite sequence of elementary row operations, each of which multiplies the determinant by a nonzero scalar (possibly 1). Thus \( \det(U) = 0 \) if and only if \( \det(A) = 0 \). Since \( U = I \) if and only if \( \det(U) \neq 0 \), we are done. \( \text{QED} \)

The following consequence must be regarded as some sort of miracle:

**Corollary.** For all \( A, B \in M_n \) we have

\[
\det(AB) = \det(A) \det(B).
\]

**Proof.** Case 1. \( A \) is invertible. Choose elementary matrices \( E_1, \ldots, E_n \) such that

\[
A = E_1 \cdots E_n.
\]

It follows from the effects of elementary row operations on determinants and the values of the determinant on elementary matrices that for every \( E, C \in M_n \), if \( E \) is elementary then

\[
\det(EC) = \det(E) \det(C).
\]

Thus

\[
\begin{align*}
\det(AB) &= \det(E_1 \cdots E_n B) \\
&= \det(E_1) \det(E_2 \cdots E_n B) \\
&= \det(E_1) \det(E_2) \det(E_3 \cdots E_n B) \\
& \cdots \\
&= \det(E_1) \cdots \det(E_n) \det(B),
\end{align*}
\]

and for the same reason

\[
\det(A) = \det(E_1) \cdots \det(E_n),
\]

so \( \det(AB) = \det(A) \det(B) \).

Case 2. \( A \) is noninvertible. Then so is \( AB \), hence

\[
\det(AB) = 0 = \det(A) \det(b).
\]

\( \text{QED} \)

**Example.** Let

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.
\]

Then

\[
\det(A) = -2 \quad \text{and} \quad \det(b) = 2.
\]

Also,

\[
AB = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix},
\]
so \( \det(AB) = -4 \), which agrees with \( \det(A) \det(B) \), as it should.

**Corollary.** If \( A \in GL_n \), then

\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]

**Proof.** This follows immediately from

\[
1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1}).
\]

**QED**

**Example.** Let

\[
A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}.
\]

Then \( \det(A) = 8 \). The reduced row echelon form of \( (A \ I) \) is

\[
\begin{pmatrix} 1 & 0 & 1/2 & -3/8 \\ 0 & 1 & 0 & 1/4 \end{pmatrix},
\]

so

\[
A^{-1} = \begin{pmatrix} 1/2 & -3/8 \\ 0 & 1/4 \end{pmatrix}.
\]

Thus \( \det(A^{-1}) = 1/8 \), which agrees with \( 1/\det(A) \), as it should.