Throughout this section, \( V \) will be a finite-dimensional inner product space.

**Theorem.** For \( u \in V \) define \( T(u): V \to \mathbb{R} \) by
\[
T(u)(x) = \langle u, x \rangle.
\]
Then:
1. \( T(u) \) is linear.
2. \( T: V \to L(V, \mathbb{R}) \) is linear and invertible.

**Proof.**
1. This follows immediately from the properties of the inner product.
2. Let \( u, v \in V \) and \( c \in \mathbb{R} \). Then for all \( x \in V \) we have
\[
T(u + v)(x) = \langle u + v, x \rangle = \langle u, x \rangle + \langle v, x \rangle = T(u)(x) + T(v)(x) = (T(u) + T(v))(x)
\]
and
\[
T(cu)(x) = \langle cu, x \rangle = c\langle u, x \rangle = cT(u)(x) = (cT(u))(x),
\]
so \( T(u + v) = T(u) + T(v) \) and \( T(cu) = cT(u) \). Thus \( T \) is linear.

To show that \( T \) is invertible, since \( \dim V = \dim L(V, \mathbb{R}) \) it suffices to show that \( \ker T = 0 \): if \( u \in \ker T \), then \( T(u) \) is the 0 element of \( L(V, \mathbb{R}) \), hence
\[
0 = T(u)(u) = \langle u, u \rangle,
\]
so \( u = 0 \).

**QED**

**Example.** Define \( S \in L(\mathbb{R}^2, \mathbb{R}) \) by
\[
S \begin{pmatrix} x \\ y \end{pmatrix} = 2x - 3y.
\]
Then for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \) we have
\[
S \begin{pmatrix} x \\ y \end{pmatrix} = \langle \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle,
\]
so in the notation of the above theorem we have
\[
S = T \begin{pmatrix} 2 \\ -3 \end{pmatrix}.
\]

**Corollary.** Let \( W \) be another finite-dimensional inner product space, and let \( T \in L(V, W) \). Then there exists a unique \( T^t \in L(W, V) \) such that for all \( x \in V \) and \( y \in W \) we have
\[
\langle T(x), y \rangle = \langle x, T^t(y) \rangle.
\]
The notation “$T^t$” in the above statement will become self-explanatory after the definition of “transpose”, which will come after the proof.

**Proof.** Let $y \in W$. Define $f \in L(V, \mathbb{R})$ by

$$f(x) = \langle T(x), y \rangle,$$

then let $T^t(y)$ be the unique vector in $V$ such that for all $x \in V$ we have

$$\langle x, T^t(y) \rangle = f(x).$$

This defines $T^t : W \to V$, and for all $x \in V$ and $y \in W$ we have

$$\langle T(x), y \rangle = \langle x, T^t(y) \rangle$$

by construction. Also, $T^t$ is the unique function with this property.

It remains to show that $T^t$ is linear: let $x, y \in W$ and $c \in \mathbb{R}$. Then for all $z \in V$ we have

$$\langle z, T^t(x+y) \rangle = \langle (T(z), x+y) = \langle T(z), x \rangle + \langle T(z), y \rangle = \langle z, T^t(x) \rangle + \langle z, T^t(y) \rangle \rangle$$

and

$$\langle z, T^t(cx) \rangle = \langle T(z), cx \rangle = c \langle T(z), x \rangle = c \langle z, T^t(x) \rangle = \langle z, cT^t(x) \rangle,$$

so we have $T^t(x+y) = T^t(x) + T^t(y)$ and $T^t(cx) = cT^t(x)$. QED

In the above corollary, $T^t$ is called the *transpose* of $T$.

**Theorem.** Let $W$ be another finite-dimensional inner product space, and define $F : L(V, W) \to L(W, V)$ by $F(T) = T^t$. Then $F$ is linear and invertible.

**Proof.** To see the linearity, let $T, S \in L(V, W)$ and $c \in \mathbb{R}$. Then for all $x \in V$ and $y \in W$ we have

$$\langle x, (T+S)^t(y) \rangle = \langle (T+S)(x), y \rangle = \langle T(x) + S(x), y \rangle = \langle T(x), y \rangle + \langle S(x), y \rangle$$

$$= \langle x, T^t(y) \rangle + \langle x, S^t(y) \rangle = \langle x, T^t(y) \rangle + \langle x, S^t(y) \rangle = \langle x, (T^t + S^t)(y) \rangle$$

and

$$\langle x, (cT)^t(y) \rangle = \langle (cT)(x), y \rangle = \langle cT(x), y \rangle = c \langle T(x), y \rangle$$

$$= c \langle x, T^t(y) \rangle = \langle x, cT^t(y) \rangle = \langle x, (cT^t)(y) \rangle,$$

so $(T + S)^t = T^t + S^t$ and $(cT)^t = cT^t$.

For the other part, since $\dim L(V, W) = \dim L(W, V)$, it suffices to show that $\ker F = \{0\}$. If $T \in \ker F$ then for all $x \in V$ and $y \in W$ we have

$$0 = \langle x, T^t(y) \rangle = \langle T(x), y \rangle,$$

so $T = 0$. QED

**Proposition.** Let $W$ and $Z$ also be finite-dimensional inner product spaces, and let $T \in L(V, W)$ and $S \in L(W, Z)$. Then:
1. \((T^t)^t = T\);
2. \((ST)^t = T^t S^t\);
3. if \(T\) is invertible, then so is \(T^t\), and 
   \[(T^t)^{-1} = (T^{-1})^t;\]
4. \(I^t = I\).

Proof. 1. For all \(x \in V\) and \(y \in W\) we have
   \[\langle (T^t)^t(x), y \rangle = \langle x, T^t(y) \rangle = \langle T(x), y \rangle,\]
   so \(T^{tt} = T\).

2. For all \(x \in V\) and \(y \in Z\) we have
   \[\langle x, (ST)^t(y) \rangle = \langle (ST)(x), y \rangle = \langle S(T(x)), y \rangle = \langle T(x), S^t(y) \rangle = \langle x, T^t(S^t(y)) \rangle = \langle x, (T^tS^t)(y) \rangle,\]
   so \((ST)^t = T^t S^t\).

3. We have
   \[(T^{-1})^t T^t = (TT^{-1})^t = I^t = I,\]
   so \(T^t\) is invertible and \((T^t)^{-1} = (T^{-1})^t\) because \(V\) and \(W\) are finite-dimensional.

4. This follows immediately from the definitions. QED

The transpose of an \(m \times n\) matrix \(A\) is the \(n \times m\) matrix \(A^t\) defined by
\[(A^t)_{ij} = A_{ji}.\]

Theorem. Let \(W\) be another finite-dimensional inner produce space, let \(E\) and \(F\) be orthonormal bases of \(V\) and \(W\), respectively, let \(T \in L(V, W)\), and let \(A\) be the matrix representing \(T\) relative to \(E\) and \(F\). Then \(A^t\) is the matrix representing \(T^t\) relative to \(F\) and \(E\).

Proof. Let \(B\) be the matrix representing \(T^t\) relative to \(F\) and \(E\). Also let \(E = \{x_1, \ldots, x_n\}\) and \(F = \{y_1, \ldots, y_m\}\). Then for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\) we have
   \[B_{ij} = \langle T^t(y_j), x_i \rangle = \langle y_j, T(x_i) \rangle = \langle T(x_i), y_j \rangle = A_{ji} = (A^t)_{ij}.\]
   QED

Corollary. 1. The function \(F: M_{m \times n} \rightarrow M_{n \times m}\) defined by \(F(A) = A^t\) is linear and invertible.
2. For all \(A \in M_{m \times n}\) we have \(A^{tt} = A\).
3. For all \(A \in M_{m \times n}\) and \(B \in M_{n \times k}\) we have \((AB)^t = B^t A^t\).
4. For all \(A \in M_n\), if \(A\) is invertible then so is \(A^t\), and 
   \[(A^t)^{-1} = (A^{-1})^t.\]
**Example.** Let $V$ be the inner product space of polynomials of degree at most 1, with inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx,
\]
and define $D \in L(V)$ by
\[
D(f) = f'.
\]

Note that the formula for this operator is
\[
D(a + bx) = b.
\]

Let’s find the transpose $D^t$ using matrices. We have seen in an earlier section that
\[
E := \left\{ \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}} x \right\}
\]
is an orthonormal basis of $V$. Our strategy is to use the matrix $A$ representing $D$ relative to this orthonormal basis. By the above theorem we know that $A^t$ represents $D^t$ relative to $E$.

First note that it is easy to find the matrix $B$ of $D$ relative to the standard basis $F := \{1, x\}$:
\[
B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

If we can find the matrix $C$ representing $D^t$ relative to $F$, then it will be easy to describe $D^t$ itself.

Let $P$ be the matrix which changes $E$-coordinates to $F$-coordinates. By the Change of Basis Theorem we have
\[
A = P^{-1}BP,
\]
so by the same theorem we have
\[
C = PA^tP^{-1}
= P(P^{-1}BP)^tP^{-1}
= PP^tB^t(P^{-1})^tP^{-1}
= PP^tB^t(PP^t)^{-1}.
\]

Now, because $F$ is the standard basis we immediately compute
\[
P = \begin{pmatrix} \sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} \end{pmatrix} = P^t,
\]
so
\[
PP^t = P^2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix},
\]
and because this matrix is diagonal the inverse is computed by just taking reciprocals of the diagonal elements:

\[
(PP^t)^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix}.
\]

Thus

\[
C = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}.
\]

From this we can immediately give a formula for the transpose operator:

\[
D^t(a + bx) = 3ax.
\]

The following result shows how the transpose allows us to convert dot products into matrix multiplication. We identify a $1 \times 1$ matrix with a scalar:

**Proposition.** For all $x, y \in \mathbb{R}^n$ we have

\[
x \cdot y = x^t y.
\]

**Proof.** This follows immediately from the definitions. \hspace{1cm} \text{QED}

$T \in L(V)$ is symmetric if $T^t = T$. Similarly, a square matrix $A$ is symmetric if $A^t = A$.

**Proposition.** The set of symmetric linear operators on $V$ is a subspace of $L(V)$.

**Proof.** Let $F$ be the linear operator on $L(V)$ defined by $F(T) = T^t$. Then the set of symmetric operators is the kernel of the operator $F - I$, hence is a subspace. \hspace{1cm} \text{QED}

**Proposition.** Let $A$ be a matrix representing $T \in L(V)$ relative to an orthonormal basis $E$. Then $T$ is symmetric if and only if $A$ is.

**Proof.** This follows immediately from the definitions since $A^t$ represents $T^t$ relative to $E$. \hspace{1cm} \text{QED}

**Lemma.** Let $T, S \in L(V)$ be symmetric, and assume that for all $x \in V$ we have

\[
\langle T(x), x \rangle = \langle S(x), x \rangle.
\]

Then $T = S$. 

Proof. Let $R = T - S$. Then $R$ is symmetric, and for all $x \in V$ we have $\langle R(x), x \rangle = 0$. Thus for all $x, y \in V$ we have

$$0 = \langle R(x + y), x + y \rangle$$
$$= \langle R(x) + R(y), x + y \rangle$$
$$= \langle R(x), x \rangle + \langle R(y), y \rangle + \langle R(x), y \rangle + \langle R(y), x \rangle$$
$$= \langle R(x), y \rangle + \langle y, R^t(x) \rangle$$
$$= \langle R(x), y \rangle + \langle R(x), y \rangle$$
$$= 2\langle R(x), y \rangle,$$

so $\langle R(x), y \rangle = 0$. It follows that $R = 0$, so $T = S$. \(\text{QED}\)

**Theorem.** Let $W$ be another finite-dimensional inner product space, and let $T \in L(V, W)$. Then:

1. $\ker T = (\text{ran } T')^\perp$;
2. $\ker T' = (\text{ran } V)^\perp$;
3. $\text{ran } T = (\ker T')^\perp$;
4. $\text{ran } T' = (\ker T)^\perp$.

**Proof.** 1. For all $x \in V$ we have:

$$x \in \ker T \iff T(x) = 0$$
$$\iff \langle T(x), y \rangle = 0 \quad \text{for all } y \in W$$
$$\iff \langle x, T'(y) \rangle = 0 \quad \text{for all } y \in W$$
$$\iff x \in (\text{ran } T')^\perp.$$

2 follows since $T = T'^t$, and then 3 and 4 follow by taking orthogonal complements. \(\text{QED}\)

**Example.** Let’s revisit an earlier example, where we found an orthonormal basis for

$$W := \text{span } \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Let’s form a matrix with column space $W$:

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
From the above corollary we have

\[ W^\perp = (\text{col } A)^\perp = \ker A^t = \ker \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \]

The reduced row echelon form of \( A^t \) is

\[
\begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.
\]

Thus a basis of \( \ker A^t \) is

\[
\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

This is a basis of \( W^\perp \), and applying the Gram-Schmidt Process to this we get the orthonormal basis

\[
\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\},
\]

which is essentially the same as our earlier result (the second vector is the negative of that in the earlier result), and this method took somewhat less work.