Let $V$ and $W$ be finite-dimensional vector spaces, let $T \in L(V,W)$, and let $b \in W$. Then $T(x) = b$ is a linear equation with unknown $x$ and constant term $b$. The solution set of the equation is $\{ x \in V : T(x) = b \}$, whose elements are the solutions of the equation. The equation is consistent if the solution set is nonempty, otherwise it is inconsistent. The equation is homogeneous if $b = 0$, in which case of course the solution set is the kernel of $T$. Reasonably enough, the equation is nonhomogeneous if it is not homogeneous.

Every homogeneous equation $T(x) = 0$ is consistent, because it has at least the trivial solution $x = 0$. All other solutions of a homogeneous equation are nontrivial.

**Proposition.**

1. $\{ b \in W : T(x) = b \text{ is consistent} \} = \text{ran } T$.
2. $T(x) = b$ is consistent for every $b \in W$ if and only if $T$ is onto.
3. $T(x) = b$ has a unique solution for every $b \in W$ if and only if $T$ is invertible.
4. A homogeneous equation $T(x) = 0$ has a nontrivial solution if and only if $T$ is not 1-1.

**Proof.** These statements follow immediately from the definitions. \( \text{QED} \)

**Proposition.** The solution set of $T(x) = b$ is a subspace of $V$ if and only if $b = 0$.

**Proof.** If $b = 0$ then the solution set is the kernel of $T$ which we know is a subspace. Conversely, suppose $b \neq 0$. If the solution set of $T(x) = b$ is empty, then it is not a subspace. If it is nonempty, let $x$ be a solution. Then $T(2x) = 2T(x) = 2b \neq b$, so $2x$ is not a solution. Thus again the solution set is not a subspace. Contrapositively, if the solution set is a subspace, then $b = 0$. \( \text{QED} \)

**Proposition.** Let $u$ be a solution of $T(x) = b$. Then the solution set is $\{ u + y : y \in \ker T \}$.

**Proof.** If $y \in \ker T$, then

$$T(u + y) = T(u) + T(y) = b + 0 = b,$$

so $u + y$ is in the solution set.

Conversely, if $T(x) = b$, then

$$T(x - u) = T(x) - T(u) = b - b = 0,$$

so $x = u + y$ with $y = x - u \in \ker T$. \( \text{QED} \)

**Corollary.** If the solution set of $T(x) = b$ is nonempty, then it has 1 element if $\ker T = \{0\}$ or infinitely many elements if $\ker T \neq \{0\}$. 
Proof. This follows from the above proposition, because a nonzero subspace is infinite. QED

The effective method of solving linear equations is to convert the equation into matrix form, solve the matrix equation, and finally interpret the result in terms of the original equation.

Example. Let $V$ be the vector space of polynomials of degree at most 2, and define $T \in L(V)$ by

$$T(a + bt + ct^2) = a + 2b + c + (b + 3c)t + (a + 2b + c)t^2.$$ 

Consider the linear equation

$$T(f) = 8 + 11t + 8t^2.$$ 

Using the standard basis of $V$, the matrix representing $T$ is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix},$$

and the coordinate vector of $8 + 11t + 8t^2$ is

$$b = \begin{pmatrix} 8 \\ 11 \\ 8 \end{pmatrix}.$$ 

Thus the linear equation is transformed into the matrix form

$$Ax = b,$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ is the unknown. We will see later that this matrix equation has the unique solution

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

so the original linear equation has the unique solution

$$f(t) = 1 + 2t + 3t^2.$$ 

We concentrate on matrix equations. If $A \in M_{m \times n}$ and $b \in \mathbb{R}^m$, then $Ax = b$ is a linear system. This is of course a special case of the above linear equations, with $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(x) = Ax$; we’ll sometimes write $L_A$ for $T$ in this situation. The linear system can also be written in the form

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1$$

$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2$$

$$\cdots$$

$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m,$$
in which it is also called a system of linear equations; to avoid confusion, sometimes the equation $Ax = b$ is called the matrix form of the linear system. $A$ is the coefficient matrix of the system, and its the entries $A_{ij}$ are the coefficients of the system. Similarly, $b$ is the constant term, its coordinates $b_i$ are the constant terms, $x$ is the unknown, and its coordinates $x_j$ are the unknowns of the system.

Since $Ax$ is the linear combination of the columns of $A$ whose coefficients are the coordinates of $x$, the range of $L_A$ is the span of the columns of $A$, and this is the column space of $A$, denoted col $A$. Other quantities associated to the linear function $L_A$ are transferred by definition directly to the matrix $A$:

1. $\ker A := \ker L_A$;
2. $\text{rank } A := \text{rank } L_A$.

The kernel of $A$ is also known as the null space of $A$, or the solution space of the homogeneous system $Ax = 0$.

Linear systems with unique solutions pose little theoretical difficulty, and inconsistent systems take only a little more effort to understand; on the other hand, we will have to give more thought to how we will handle systems with infinitely many solutions.

Observe that the coefficients and the constant terms carry all the information of the linear system; in fact, we will solve the system using only the matrices $A$ and $b$. Before indicating precisely how this is done, we give a few preliminaries in a more general context (because it will be useful later): let $A \in M_{m \times n}$ and $C \in M_{m \times k}$, and write these matrices in terms of their columns:

$$A = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 & \cdots & c_k \end{pmatrix},$$

where $a_1, \ldots , a_n, c_1, \ldots , c_k \in \mathbb{R}^m$. Then

$$\begin{pmatrix} A & C \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_k \end{pmatrix} \in M_{m \times (n+k)}$$

is a partitioned matrix. If we also have $D \in M_{l \times m}$, then the properties of matrix multiplication imply that

$$D \begin{pmatrix} A & C \end{pmatrix} = \begin{pmatrix} DA & DC \end{pmatrix}.$$

Back to linear systems: the partitioned matrix

$$\begin{pmatrix} A & b \end{pmatrix}$$

is the augmented matrix of the linear system $Ax = b$.

**Example.** The augmented matrix of the system

\[
\begin{align*}
2x_1 - 3x_2 &= 5 \\
5x_1 + x_2 &= 0 \\
-x_2 &= 7
\end{align*}
\]
is
\[
\begin{pmatrix}
2 & -3 & 5 \\
5 & 1 & 0 \\
0 & -1 & 7
\end{pmatrix}.
\]
The linear system whose augmented matrix is
\[
\begin{pmatrix}
0 & 1 & 0 & 3 \\
3 & -1 & 4 & 2 \\
1 & 1 & 2 & 0
\end{pmatrix}
\]
is
\[
x_2 = 3 \\
3x_1 - x_2 + 4x_3 = 2 \\
x_1 + x_2 + 2x_3 = 0.
\]

If \( A, C \in M_{m \times n} \), then linear systems \( Ax = b \) and \( Cx = d \) are \textit{equivalent} if they have the same solution set. The general strategy for solving linear systems is to find an equivalent system whose solution is obvious. The method we will introduce, called \textit{row reduction}, uses the observation recorded in the following proposition.

**Proposition.** If \( D \in GL_m \), \( A \in M_{m \times n} \), and \( b \in \mathbb{R}^m \), then the systems \( Ax = b \) and \( DAx = Db \) are equivalent.

**Proof.** Exercise. \( \square \)

**Corollary.** If \( A \in GL_n \), then for every \( b \in \mathbb{R}^n \) the system \( Ax = b \) has the unique solution \( x = A^{-1}b \).

What if \( A \) is square and noninvertible?

**Proposition.** If \( A \in M_n \) is noninvertible, then for each \( b \in \mathbb{R}^n \) the system \( Ax = b \) is either inconsistent or has infinitely many solutions, and both possibilities occur.

**Proof.** Since \( A \) is square and noninvertible, the linear operator \( L_A \) is neither 1-1 nor onto. Since it is linear and not 1-1, there is no \( b \in \mathbb{R}^n \) for which the system \( Ax = b \) has a unique solution. Since it is not onto, there exists \( b \in \mathbb{R}^n \) for which \( Ax = b \) is inconsistent. Finally, the homogeneous system \( Ax = 0 \) is consistent, hence has infinitely many solutions. \( \square \)

**Example.** Consider the linear system
\[
\begin{align*}
2x_1 + x_3 &= 4 \\
3x_1 - x_2 + x_3 &= 5.
\end{align*}
\]
The coefficient matrix is
\[
A = \begin{pmatrix}
2 & 0 & 1 \\
3 & -1 & 1
\end{pmatrix}.
\]
the constant term is
\[ b := \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \]
and the augmented matrix is
\[ (A \; b) = \begin{pmatrix} 2 & 0 & 1 & 4 \\ 3 & -1 & 1 & 5 \end{pmatrix}. \]
Letting
\[ D = \begin{pmatrix} 1/2 & 0 \\ 3/2 & -1 \end{pmatrix}, \]
we have
\[ D (A \; b) = \begin{pmatrix} 1 & 0 & 1/2 & 2 \\ 0 & 1 & 1/2 & 1 \end{pmatrix}. \]
The system associated to this latter augmented matrix is
\[
\begin{align*}
    x_1 + (1/2)x_3 &= 2 \\
    x_2 + (1/2)x_3 &= 1.
\end{align*}
\]
This system is equivalent to the given one, and we can solve it easily: first note that it is obvious that

\[ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \]
is a solution. Next, the coefficient matrix of this new system is
\[ C := \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{pmatrix}, \]
which has rank 2 (because its column space is a subspace of \( \mathbb{R}^2 \) containing the standard basis, namely the 1st and 2nd columns of \( C \)). Thus the nullity of \( C \) is 1 (because \( C \) has 3 columns and rank 2). The associated homogeneous system
\[
\begin{align*}
    x_1 + (1/2)x_3 &= 0 \\
    x_2 + (1/2)x_3 &= 0
\end{align*}
\]
can be immediately solved for \( x_1 \) and \( x_2 \) in terms of \( x_3 \):
\[
\begin{align*}
    x_1 &= -(1/2)x_3 \\
    x_2 &= -(1/2)x_3.
\end{align*}
\]
Since the kernel is 1-dimensional, to get a basis we need a single nonzero solution of the homogeneous system, which we find by choosing a nonzero value for \( x_3 \) and using the above
equations to compute the corresponding values of $x_1$ and $x_2$. For example, if we take $x_3 = 2$, then $x_1 = x_2 = -1$, so

$$\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

is a basis for the kernel. Thus the kernel is

$$\left\{ t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\},$$

hence the solution set of the system with augmented matrix $(DA \ Db)$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}.$$ 

Since this system is equivalent to the original system, we have solved the linear system with augmented matrix $(A \ b)$.

What properties of the augmented matrix $(DA \ Db)$ made finding this solution so easy? The columns of the coefficient matrix $C$ included the standard basis vectors $e_1, e_2$, and the other columns in the augmented matrix were in the span of $e_1, e_2$. The columns containing $e_1, e_2$ were the 1st and the 2nd, and the corresponding unknowns were thus $x_1, x_2$. This allowed us to immediately find a particular solution of the system, and also solve the associated homogeneous system for the unknowns $x_1, x_2$ in terms of the other unknowns ($x_3$ in this example).

Two questions arise:

1. How do we find a suitable invertible matrix $D$?
2. Is the resulting transformed augmented matrix $(DA \ Db)$ unique?

We will answer Question 1 a little later. Regarding Question 2, the answer is: no! (However, we will see in a little while that we can make it unique by requiring it to satisfy some further properties.) Let’s illustrate the nonuniqueness by giving two other invertible matrices $D_1$ and $D_2$, each of which transforms the augmented matrix into one with the same properties as above: let

$$D_1 = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$ 

We have

$$D_1 (A \ b) = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}.$$
This time the coefficient matrix is
\[ C_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \]
which contains \( e_1, e_2 \) as the 1st and 3rd columns, and the other columns in the augmented matrix are in the span of \( e_1, e_2 \). The associated system is
\[
\begin{align*}
  x_1 - x_2 &= 1 \\
  2x_2 + x_3 &= 2,
\end{align*}
\]
and we immediately see that
\[
\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}
\]
is a solution. The associated homogeneous system
\[
\begin{align*}
  x_1 - x_2 &= 0 \\
  2x_2 + x_3 &= 0
\end{align*}
\]
can be solved for \( x_1, x_3 \) in terms of \( x_2 \):
\[
\begin{align*}
  x_1 &= x_2 \\
  x_3 &= -2x_2.
\end{align*}
\]
The same sort of reasoning as above shows that the kernel is 1-dimensional, and we get a basis
\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}
\]
by taking \( x_2 = 1 \), so the kernel is
\[
\left\{ t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]
Thus the solution set of the system is
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]
Using \( D_2 \) instead, we have
\[
D_2 (A \ b) = \begin{pmatrix} 2 & 0 & 1 & 4 \\ -1 & 1 & 0 & -1 \end{pmatrix},
\]
with associated system
\[
\begin{align*}
2x_1 + x_3 &= 4 \\
-x_1 + x_2 &= -1,
\end{align*}
\]
so
\[
\begin{pmatrix}
0 \\
-1 \\
4
\end{pmatrix}
\]
is a solution, and we solve the associated homogeneous system
\[
\begin{align*}
2x_1 + x_3 &= 0 \\
-x_1 + x_2 &= 0,
\end{align*}
\]
for \(x_2, x_3\) in terms of \(x_1\):
\[
x_2 = x_1 \\
x_3 = -2x_1,
\]
so the kernel is
\[
\left\{ t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\},
\]
hence the solution set of the given system is
\[
\left\{ \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

Let’s compare and contrast the three different solutions of the linear system of the preceding example: note first of all that, unsurprisingly (because the kernel is 1-dimensional), the bases for the kernel are all parallel. Also, in each case there were two columns of the resulting coefficient matrix which contained the standard basis vectors \(e_1, e_2\), but that these columns varied from solution to solution. This gives us a way to single out a preferred method: if the coefficient matrix \(A\) is \(m \times n\), we impose the restriction that the transformed augmented matrix \((DA\ Db)\) should have the property that there exist natural numbers
\[
1 \leq j_1 < j_2 < \cdots < j_r \leq n
\]
such that the standard basis vectors
\[
e_1, e_2, \ldots, e_r
\]
appear in columns \(j_1, j_2, \ldots, j_r\) of \((DA\ Db)\), and every other column is in the span of the columns preceding it. In the above example, this singles out the first solution method.

This is the start of a topic unto itself, which we explore in the next section.