Be sure to review the “Preliminaries on sets”, the “Preliminaries on numbers”, and the “Preliminaries on functions”.

The Euclidean plane $\mathbb{R}^2$ and its properties form a paradigm for a very general type of mathematical structure which has wide applications. Here is the abstract definition:

**Vector spaces.** A *vector space* is a set $V$ equipped with two operations:

- **Addition:** for all $x, y \in V$ there is an element $x + y \in V$, called the *sum* of $x$ and $y$;
- **Scalar multiplication:** for all $c \in \mathbb{R}$ and $x \in V$ there is an element $cx \in V$, called the *scalar multiple* of $x$ by $c$,

which satisfy the following properties:

1. For all $x, y \in V$ we have
   \[ x + y = y + x. \]
2. For all $x, y, z \in V$ we have
   \[ (x + y) + z = x + (y + z). \]
3. There exists $0 \in V$ such that for all $x \in V$ we have
   \[ x + 0 = x. \]
4. For all $x \in V$ there exists $-x \in V$ such that
   \[ x + (-x) = 0. \]
5. For all $c, d \in \mathbb{R}$ and $x \in V$ we have
   \[ (c(dx)) = (cd)x. \]
6. For all $c \in \mathbb{R}$ and $x, y \in V$ we have
   \[ c(x + y) = cx + cy. \]
7. For all $c, d \in \mathbb{R}$ and $x \in V$ we have
   \[ (c + d)x = cx + dx. \]
8. For all $x \in V$ we have
   \[ 1x = x. \]
The elements of $V$ are called *vectors*, and (as mentioned before) the elements of $\mathbb{R}$ are called *scalars*.

In 3 above, we use the notation $0$ to signify that we have singled out a particular vector with the indicated property. But just as for $\mathbb{R}^2$, this vector is unique: if we have $x + 0' = x$ for all $x \in V$, then

$$0 = 0 + 0' = 0' + 0 = 0'.$$

0 is called the *zero vector* of $V$.

Similarly, in 4 above we use the notation $-x$ to indicate a vector with the indicated property, but we make no claim of uniqueness; in fact, it follows from the definition of vector space that this vector is uniquely determined by $x$ (exercise). $-x$ is called the *negative* of $x$.

As we mentioned in the discussion of $\mathbb{R}^2$, the reason for listing the above 8 properties is that they imply all other properties of the operations of addition and scalar multiplication. For example:

**Proposition.** Let $V$ be a vector space.

1. For all $x, y, z \in V$, if $x + y = x + z$ then $y = z$.
2. For all $c \in \mathbb{R}$ we have $c0 = 0$ (here 0 is the zero vector in $V$).
3. For all $x \in V$ we have $0x = 0$ (here the first 0 is in $\mathbb{R}$ and the second is in $V$).
4. For all $x \in V$ we have $-x = (-1)x$.

Subtraction of vectors is defined by

$$x - y := x + (-y) \quad \text{for } x, y \in V.$$ 

In vector expressions containing both addition and scalar multiplication, scalar multiplication takes precedence (*i.e.*, is performed first) unless parentheses override; for example, if $a, b \in \mathbb{R}$ and $x, y \in V$ then $ax + by$ is the sum of the vectors $ax$ and $by$.

Here are some other examples of vector spaces:

**Example.** This example generalizes $\mathbb{R}^2$. Let $n \in \mathbb{N}$, and put

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}.$$ 

For each $i = 1, \ldots, n$, $x_i$ is the *ith coordinate* of $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. 
Define addition and scalar multiplication in $\mathbb{R}^n$ by
\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} + \begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{pmatrix} = \begin{pmatrix}
  x_1 + y_1 \\
  \vdots \\
  x_n + y_n
\end{pmatrix}
\]
\[
c\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  cx_1 \\
  \vdots \\
  cx_n
\end{pmatrix}.
\]
The zero vector in $\mathbb{R}^n$ is
\[
\begin{pmatrix}
  0 \\
  \vdots \\
  0
\end{pmatrix},
\]
and the negative of a vector is given by
\[
-\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  -x_1 \\
  \vdots \\
  -x_n
\end{pmatrix}.
\]

**Example.** Let $m, n \in \mathbb{N}$. An $m \times n$ matrix is a rectangular array of real numbers with $m$ rows and $n$ columns, so that if $A$ is an $m \times n$ matrix then $A$ has the form
\[
A = \begin{pmatrix}
  A_{11} & A_{12} & \cdots & A_{1n} \\
  A_{21} & A_{22} & \cdots & A_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  A_{m1} & A_{m2} & \cdots & A_{mn}
\end{pmatrix}.
\]

$A_{ij}$ is called the $ij$-entry of $A$. We also write $A = (A_{ij})$. The set of all $m \times n$ matrices is denoted by $M_{m \times n}$. Also, we write $M_n = M_{n \times n}$.

Define addition and scalar multiplication in $M_{m \times n}$ as follows: if $A, B \in M_{m \times n}$, then $C = A + B$ is the $m \times n$ matrix with $ij$-entry
\[
C_{ij} = A_{ij} + B_{ij},
\]
and if $c \in \mathbb{R}$ then $D = cA$ is the $m \times n$ matrix with $ij$-entry
\[
D_{ij} = cA_{ij}.
\]
We can write $(A_{ij}) + (B_{ij}) = (A_{ij} + B_{ij})$ and $(cA_{ij}) = (cA_{ij})$.

Then zero vector is the $m \times n$ matrix with all entries equal to 0, and negatives are given by
\[
-(A_{ij}) = (-A_{ij}).
\]

Note that $M_{n \times 1} = \mathbb{R}^n$ as vector spaces.
Example. A polynomial is an expression of the form
\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \]
where \( n \) is a nonnegative integer and each \( a_i \in \mathbb{R} \). If \( a_n \neq 0 \) then \( f(x) \) has degree \( n \). In particular, nonzero real numbers are polynomials of degree 0. The degree of the zero polynomial is defined to be \(-1\) for technical reasons. The \( a_i \)'s are the coefficients of \( f(x) \).

Define addition and scalar multiplication of polynomials by:
\[
(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n
\]
and
\[
c(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = ca_0 + ca_1 x + ca_2 x^2 + \cdots + ca_n x^n.
\]
Note that to add two polynomials, if one has smaller degree than the other then extra terms with coefficient 0 must be (tacitly) added to use the above definition.

The zero polynomial is the scalar 0, and negatives are given by
\[
-(a_0 + a_1 x + \cdots + a_n x^n) = -a_0 - a_1 x - \cdots - a_n x^n.
\]

Not only is the set of all polynomials a vector space, but we get other vector spaces by restricting the degree: for each \( n \in \mathbb{N} \) we can (and will) consider the vector space of all polynomials of degree at most \( n \).

Example. Let \( X \) be a set. Then the set of all functions from \( X \) to \( \mathbb{R} \) is a vector space with the pointwise operations: if \( f, g : X \to \mathbb{R} \) and \( c \in \mathbb{R} \) then addition and scalar multiplication are defined by
\[
(f + g)(x) := f(x) + g(x)
\]
\[
(cf)(x) := cf(x).
\]
Many important vector spaces consist of functions, for example the vector space of all continuous functions on an interval \([a, b]\), or the vector space of all differentiable functions on an interval \([a, b]\).