Exercise 1. In each part, decide whether $u$ is an eigenvector of $T$, and if so determine the associated eigenvalue:

(a) $T = L_A$ with $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) same $T$ as part (a), $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(c) same $T$ as part (a), $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(d) $T = L_A$ with $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(e) same $T$ as part (d), $u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(f) $T \in L(M_2)$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $u = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(g) same $T$ as part (f), $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Exercise 2. Define $T \in L(\mathbb{R}^n)$ by

$$T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}.$$ 

Find an eigenvector with associated eigenvalue 1.

Exercise 3. Let $A \in M_n$ and $\lambda \in \mathbb{R}$, and assume that the entries in each row of $A$ sum to $\lambda$. Prove that $\lambda$ is an eigenvalue of $A$.

Exercise 4. Let $T \in L(\mathbb{R}^2)$ be rotation by $\pi/3$. Is $T$ diagonalizable? Why or why not?

Exercise 5. Let $V$ be a 3-dimensional vector space, $T \in L(V)$, and $\{x_1, x_2, x_3\}$ a basis of $V$. Assume that $x_1$ and $x_2$ are eigenvectors of $T$ with associated eigenvalue 2, and $x_3$ is an eigenvector with associated eigenvalue $-5$.

(a) What is the matrix representing $T$ relative to the basis $\{x_1, x_2, x_3\}$?
(b) What is the matrix representing $T$ relative to the basis $\{x_1, x_3, x_2\}$?
Exercise 6. Let
\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \]

(a) Compute \( AP \).

(b) Use the result of part (a) to determine whether \( P \) diagonalizes \( A \).

Exercise 7. Let \( V \) be the vector space of polynomials of degree at most 1, let \( T \in L(V) \), let \( E \) be the basis \( \{1 + x, 2 - x\} \) of \( V \), let \( A \) be the matrix representing \( T \) relative to \( E \), and let
\[ P = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}. \]
Assume that \( P^{-1}AP \) is diagonal. Find a basis of \( V \) which diagonalizes \( T \).

Exercise 8.

(a) Prove that every upper triangular matrix has an eigenvector.

(b) Prove that every lower triangular matrix has an eigenvector.

Exercise 9. Let \( V \) be a vector space and \( T \in L(V) \), and let \( x \) be an eigenvector of \( T \) with associated eigenvalue \( \lambda \).

(a) Prove using induction that if \( n \in \mathbb{N} \) then \( x \) is an eigenvector of \( T^n \) with associated eigenvalue \( \lambda^n \).

(b) Prove that if \( T \) is invertible then \( x \) is an eigenvector of \( T^{-1} \) with associated eigenvalue \( 1/\lambda \).

Exercise 10. Use part (a) of the preceding exercise to prove that if there exists \( n \in \mathbb{N} \) such that \( T^n = 0 \) then every eigenvalue of \( T \) is 0.