Name:

- Be sure to show all your work!
- No books, notes or calculators of any kind are permitted on the test.
- Write each solution in the space provided, not on scratch paper.
- If you need more room, write on the back of the page. If you still need more room, ask for scratch paper.
- Be sure to give reasons for your answers (except where explicitly told otherwise)!
- Points will probably be taken off if you do not give a clear indication of the method and write your solution neatly, in order, and clearly indicate your final answer.
- Points may be taken off for a correct answer with a mistake in the supporting work.
1. In this problem, only the answer will be graded; you do not need to show any work!

(a) Find $||x||$ if $x = (2 + 3i, -1) \in \mathbb{C}^2$.

(b) What is $UU^H$ if $U$ is a unitary matrix?

(c) Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in an inner product space $V$. If $||\mathbf{x}|| = 2$ and $||\mathbf{y}|| = 3$, what is the largest possible value of $|\langle \mathbf{x}, \mathbf{y} \rangle|$?

(d) Exactly one of the matrices $A = \begin{pmatrix} 2 + 3i & 0 \\ 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 7 - 4i \\ 7 + 4i & 3 \end{pmatrix}$, or $C = \begin{pmatrix} 2 & 7 - 4i \\ 7 - 4i & 3 \end{pmatrix}$ is Hermitian. Which is it?

(e) Let $A$ be a $5 \times 9$ matrix with rank 3, and let $S = N(A)$. What is the dimension of $S^\perp$?

(f) What are the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 6 \end{pmatrix}$?
2. (a) Find $\cos \theta$ if $\theta$ is the angle between the vectors $\mathbf{x} = (1, 0, 1, 1)$ and $\mathbf{y} = (1, 2, 2, 2)$ in $\mathbb{R}^4$ (with the standard inner product).

(b) Let $\mathbb{R}^2$ have the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{2} x_i y_i w_i$, where $\mathbf{w} = (3, 1)$ is the weight vector. If $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (2, -3)$, determine whether $\mathbf{x}$ and $\mathbf{y}$ are orthogonal with respect to this inner product.

(c) Let $P_2$ have the inner product $\langle p, q \rangle = \int_0^1 p(x) q(x) \, dx$. Find the distance between $p(x) = x$ and $q(x) = 1$. 
3. (a) Let \( A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ -2 & 2 \end{pmatrix} \) and \( b = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \). Find the normal equations (also called the normal system) of the system \( Ax = b \).

(b) Let \( A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \). Given that \( \bar{x} = \begin{pmatrix} 6 \\ \frac{11}{11} \\ \frac{8}{11} \end{pmatrix} \) is the least squares solution of the system \( Ax = b \), find the vector in the column space of \( A \) which is closest to the vector \( b \).
4. Let $A$ be a $4 \times 4$ matrix which is real (that is, its entries are all real numbers) and symmetric. Given that the eigenvalues of $A$ are 1 and 2, and that the eigenspace associated with $\lambda = 1$ has basis \[
\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}
\] and the eigenspace associated with $\lambda = 2$ has basis \[
\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\},
\] find an orthogonal matrix $U$ that diagonalizes $A$. 
5. Let \( A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \).

(a) Find the eigenvalues of \( A \).

(b) Find a matrix \( X \) such that \( D = X^{-1}AX \) is diagonal.

(c) Find the matrix \( D \) referred to in part (b).