MAT 342 TEST 3  
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- Points will probably be taken off for a correct answer without reasons or a clear indication of the method.
- Points may be taken off for a correct answer with a mistake in the supporting work.
- No books, notes or calculators of any kind are permitted on the test.
- Clearly write your solution, in order, and indicate your final answer.
- Write each solution on the same page as the problem, not on scratch paper.
- If you need more room, write on the back of the page.
- Time limit: 1.5 hours.
1. In each part of this problem, only the answer will be graded:

(a) Let $V$ be an inner product space, let $x \in V$, let $S$ be a subspace of $V$, let $p$ be the projection of $x$ onto $S$, and let $y$ be any vector in $S$. What is $\langle x - p, y \rangle$?

(b) If $Q$ is an orthogonal matrix, what is $Q^T Q$?

(c) If $x$ and $y$ are orthogonal unit vectors in an inner product space $V$, what is $\langle 2x + 3y, -x + 3y \rangle$?

(d) Let $Q$ be an orthogonal $n \times n$ matrix and $x \in \mathbb{R}^n$. If $\|x\| = 2$, what is $\|Qx\|$?

(e) Let $x$ and $y$ be vectors in an inner product space $V$. If $\|x\| = 3$ and $\|y\| = 4$, how big can $|\langle x, y \rangle|$ be?

(f) Let $S$ be a subspace of an inner product space $V$. If $\dim S = 5$ and $\dim V = 8$, what is $\dim S^\perp$?
2. Let \( \mathbb{R}^{2 \times 2} \) have the inner product

\[
\langle A, B \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} b_{ij}.
\]

If

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix},
\]

find:

(a) \( \cos \theta \), where \( \theta \) is the angle between \( A \) and \( B \);

(b) the vector projection of \( A \) onto \( B \);

(c) the distance between \( A \) and \( B \).
3. Let \( x_1 = (1,1,1,5), x_2 = (1,2,0,7) \in \mathbb{R}^4 \), and let \( S = \text{Span}(x_1, x_2) \). Given that the reduced row echelon form of the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 5 \\
1 & 2 & 0 & 7
\end{pmatrix}
\]

is

\[
\begin{pmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -1 & 2
\end{pmatrix},
\]

find a basis for \( S^\perp \).
4. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Use the Gram-Schmidt Process to find an orthonormal basis of $R(A)$.

(b) Use the computations of part (a) to find an orthogonal matrix $Q$ and an upper triangular invertible matrix $R$ such that $A = QR$. 
5. Find all least squares solutions of the system

\[
\begin{align*}
  x_1 + x_2 &= 1 \\
  2x_1 - 3x_2 &= 1 \\
  -x_1 - x_2 &= 2.
\end{align*}
\]
6. Let $C[-1, 1]$ have the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx,$$

and let $S$ be the subspace of $C[-1, 1]$ spanned by the orthonormal set

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} \right\}.$$

Find the function $p$ in $S$ which is closest to $x^2$. 