MAT 342 TEST 2

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- There will be **no** partial credit awarded unless the supporting work is clearly given!
- No books, notes or calculators of any kind are permitted on the test.
- All answers must be **clearly** written in **final** form!
- Write each solution on the **same page as the problem**!
- If you need more room, write on the back of the page.
- **Time limit: 1.5 hours.**
1. Let
\[
A = \begin{pmatrix}
1 & 2 & 1 & -2 & 1 \\
-1 & -2 & 1 & -4 & 2 \\
0 & 0 & 2 & -6 & 3 \\
2 & 4 & 1 & -1 & 5
\end{pmatrix}.
\]
Given that the reduced row echelon form of \( A \) is
\[
\begin{pmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
find:

(a) a basis for the column space of \( A \);

(b) a basis for the row space of \( A \);

(c) a basis for the null space of \( A \).
2. Suppose $A$ is a $4 \times 7$ matrix with rank 3. Find:

(a) the dimension of the column space of $A$;

(b) the dimension of the row space of $A$;

(c) the nullity of $A$. 
3. Let \( L : V \rightarrow V \) be a linear operator, and let \( E \) and \( F \) be two bases for \( V \). Suppose the matrix representing \( L \) with respect to \( E \) is
\[
A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}
\]
and that the transition matrix from \( F \) to \( E \) is
\[
S = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}.
\]
Find the matrix \( B \) which represents \( L \) with respect to \( F \).
4. Let
\[ E = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \\ 5 \end{pmatrix} \right\} \]

be two bases for \( \mathbb{R}^2 \). Find a matrix \( S \) such that \( S[v]_E = [v]_F \) for all \( v \in \mathbb{R}^2 \).
5. Let $L: P_2 \to P_3$ be the linear transformation defined by

$$L(a + bx) = a + b - 2ax + (a - b)x^2,$$

and let $E = \{1 + x, -2x\}$ be a basis for $P_2$ and $F = \{1, 1 + x, 1 + x + x^2\}$ be a basis for $P_3$. Find a matrix $A$ such that $[L(p)]_F = A[p]_E$ for all $p \in P_2$. 

6. Let \( L : V \rightarrow W \) be a linear transformation, and let \( v_1, \ldots, v_n \) be vectors in \( V \). Prove that if the vectors \( L(v_1), \ldots, L(v_n) \) in \( W \) are linearly independent, then \( v_1, \ldots, v_n \) are also linearly independent. Hint: \( L(0) = 0 \).