FINDING THE MATRIX OF A LINEAR MAP

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I want to show you a point of view regarding the problem of finding a matrix representation of a linear map. The main idea is to coordinate! This is just one illustration of how matrix techniques can be used to handle all questions in finite-dimensional linear algebra. Remember: once we choose a basis $E$ of an $n$-dimensional vector space $V$, all problems in $V$ can be translated into problems in $\mathbb{R}^n$.

Given a linear map $L: V \to W$ and bases $E$ of $V$ and $F$ of $W$, we get the following diagram of linear maps:

\[
\begin{array}{c}
V \\
\downarrow L \\
W \\
E \\
\downarrow F \\
\mathbb{R}^n \\
\downarrow A \\
\mathbb{R}^m,
\end{array}
\]

where $\dim V = n$, $\dim W = m$, the vertical arrows represent the identifications afforded by the choice of bases, and in the bottom arrow $A$ is the matrix representing $L$ with respect to $E$ and $F$. Recall that this means that if $x \in V$ and the coordinate vector of $x$ with respect to $E$ is written $x_E$ (note that we’re simplifying the notation a little from the book), then

$$L(x)_F = Ax_E,$$

where of course the left hand side is the coordinate vector of $L(x)$ with respect to $F$.

The problem is to find the matrix $A$ given $L$, $E$, and $F$. The point of view we take here is that, if we can find some matrix for $L$ (with respect to some bases for $V$ and $W$), then finding the desired matrix $A$ becomes a change-of-basis problem. So that we can talk about things, let’s say we have a matrix $B$ representing $L$ with respect to bases $G$ of $V$ and $H$ of $W$. Then we get the following (somewhat elaborate) diagram of linear maps:

\[
\begin{array}{c}
\mathbb{R}^n \\
\downarrow A \\
\mathbb{R}^m \\
E \\
\downarrow L \\
\downarrow F \\
\mathbb{R}^n \\
\downarrow B \\
\mathbb{R}^m,
\end{array}
\]

where in the vertical arrows $S$ and $T$ are the transition matrices from $E$ to $G$ and $F$ to $H$. Recall that this means that if $x \in V$ then

$$S x_E = x_G,$$

and similarly if $y \in W$ then

$$T y_F = y_H.$$  

Of course, $B$ does the same kind of job that $A$ does:

$$L(x)_H = B x_G \quad \text{for } x \in V.$$ 

Now, the diagram tells us that if we start at the upper left hand corner we can take several paths to the lower right hand corner, and we should get the same result every time. In particular, going to the right using $A$ and then going down using $T$ gives the same result as first going down using $S$ and then to the right using $B$:

$$TA = BS.$$
This gives a very precise connection between the matrices $A$ and $B$; it is slightly more general than the “similarity” we studied in the textbook, but uses the same ideas.

**Now for the beauty of the method:** a lot of the time we can find bases $G$ and $H$ — typically “standard bases” for which the matrices $B$, $S$, and $T$ are easy to find. Then the relation $TA = BS$ lets us find $A$ by row-reducing an appropriate augmented matrix:

\[
(T \mid BS) \to (I \mid A).
\]

Of course, we could write the relation as $A = T^{-1}BS$, but this is not helpful for the computation: it makes us first find $T^{-1}$ and then multiply it by $BS$; the row-reduction is less work.

To summarize: once we find “nice” $G$ and $H$, so that $S$, $T$, and $B$ come easily, the problem of finding $A$ is translated into a matrix problem — it has been “coordinatized”!

**Example.** Let’s do Problem 5 from Test 2. The situation was:

\[
\begin{array}{c|c}
\mathbb{R}^2 & \mathbb{R}^3 \\
E & A \\
S & F \\
B & \mathbb{R}^2
\end{array}
\]

\[
L(a + bx) = a + b - 2ax + (a - b)x^2
\]
\[
E = \{1 + x, -2x\}
\]
\[
F = \{1, 1 + x, 1 + x + x^2\}.
\]

For $G$ and $H$ we take the standard bases:

\[
G = \{1, x\}
\]
\[
H = \{1, x, x^2\}.
\]

Then we see easily that

\[
B = \begin{pmatrix}
1 & 1 \\
-2 & 0 \\
1 & -1
\end{pmatrix}
\]
\[
S = \begin{pmatrix}
1 & 0 \\
1 & -2
\end{pmatrix}
\]
\[
T = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Therefore the desired matrix $A$ satisfies

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
A = \begin{pmatrix}
1 & 1 \\
-2 & 0 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & -2
\end{pmatrix} = \begin{pmatrix}
2 & -2 \\
-2 & 0 \\
0 & 2
\end{pmatrix}.
\]

We solve for $A$ by row-reduction:

\[
\begin{pmatrix}
1 & 1 & 1 & 2 & -2 \\
0 & 1 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 & 2
\end{pmatrix} \to \begin{pmatrix}
1 & 1 & 0 & 2 & -4 \\
0 & 1 & 0 & -2 & -2 \\
0 & 0 & 1 & 0 & 2
\end{pmatrix}
\]
\[
\to \begin{pmatrix}
1 & 0 & 0 & 4 & -2 \\
0 & 1 & 0 & -2 & -2 \\
0 & 0 & 1 & 0 & 2
\end{pmatrix}.
\]
Therefore

\[ A = \begin{pmatrix} 4 & -2 \\ -2 & -2 \\ 0 & 2 \end{pmatrix} \] .

Of course, this matrix method works in the special case where \( V = \mathbb{R}^n \) and \( W = \mathbb{R}^m \), and then \( B \) would be the standard matrix for \( L \).

But here's a good thing: it also works for finding transition matrices! Suppose \( E \) and \( F \) are two bases for \( V \), and we want the transition matrix \( A \) from \( E \) to \( F \) (and I call it \( A \) here because of how it will appear in our diagram). Important fact: \( A \) is the matrix representing the identity operator \( I: V \to V \)! We can take some "nice" basis \( G = H \), and now our diagram becomes:

![Diagram](image)

where in the bottom arrow the identity matrix \( I \) represents the identity operator with respect to the basis \( G \). This is a general method, and in each case the choice of the "nice" basis \( G \) must be made. If \( V \) happens to have some "standard basis" then it's probably best to use it.

*Example.* Let's do Problem 4 from Test 2. The situation was:

![Diagram](image)

where

\[ E = \left\{ \left( \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \right\} \quad \text{and} \quad F = \left\{ \left( \begin{array}{c} 1 \\ 1 \\ 4 \\ 5 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 5 \\ 2 \end{array} \right) \right\} . \]

For \( G \) we take the standard basis for \( \mathbb{R}^2 \), and we easily get

\[ S = \left( \begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array} \right) \quad \text{and} \quad T = \left( \begin{array}{cc} 1 & 4 \\ 1 & 5 \end{array} \right) . \]

So, the desired transition matrix \( A \) (and again I'm calling it \( A \) here because of our method, although it was called "\( S \)" in the test itself) satisfies

\[ \left( \begin{array}{cc} 1 & 4 \\ 1 & 5 \end{array} \right) A = I \left( \begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array} \right) = \left( \begin{array}{cc} 2 & 1 \\ 3 & 2 \end{array} \right) , \]

and we solve for \( A \) by row-reduction:

\[ \left( \begin{array}{cccc} 1 & 4 & 2 & 1 \\ 1 & 5 & 3 & 2 \end{array} \right) \to \left( \begin{array}{cccc} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right) \to \left( \begin{array}{cccc} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 1 \end{array} \right) , \]

giving

\[ A = \left( \begin{array}{cc} -2 & -3 \\ 1 & 1 \end{array} \right) . \]