MAT 300 Notes on Section 3.6
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To show $\exists! x P(x)$:

Existence:
(give a self-contained proof of $\exists x P(x)$)

Uniqueness:
Let $y$ be arbitrary.
Assume $P(y)$.

Thus $y = x$.

Alternative strategy for proving uniqueness (not so common, in my experience, contrary to the textbook author’s assertions):

Let $y$ and $z$ be arbitrary.
Assume $P(y)$ and $P(z)$.

Thus $y = z$.

The main difference is that, in the 2nd alternative, the proof of uniqueness is completely independent of the proof of existence.
To use $\exists! x P(x)$:

Let $x$ be the unique object such that $P(x)$.

:$(\text{some } y \text{ such that } P(y) \text{ appears})$
Then $y = x$.

Alternatively:

$(\text{some } y \text{ and } z \text{ appear such that } P(y) \text{ and } P(z))$
Then $y = z$. 
To prove a the statements in a list are equivalent:

**Theorem 1.** The following are equivalent:

1. $P$.
2. $Q$.
3. $R$.

*Proof.* First assume $P$.

\[ \vdash \]
Therefore $Q$.
Next, assume $Q$.

\[ \vdash \]
Therefore $R$.
Finally, assume $R$.

\[ \vdash \]
Therefore $P$. \qed

Actually, there could be any number of equivalent statements. The main thing is that all you have to prove is a chain of implications starting and ending with one of the statements, making sure to hit every statement along the way. In particular, you don’t have to prove $1 \leftrightarrow 2$, then $2 \leftrightarrow 3$, etc.