MAT 300 Notes on Section 6.4
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I didn’t have time to do all the examples in class, so here I’ll do the remaining example:

**Theorem** (Well-Ordering Principle). *Every nonempty subset of \( \mathbb{N} \) has a smallest element.*

**Proof.** Let \( A \subseteq \mathbb{N} \). Assume that \( A \) does not have a smallest element. We’ll deduce that \( A = \emptyset \), so by contrapositive we’ll be able to conclude that if \( A \neq \emptyset \) then \( A \) has a smallest element. To show that \( A \) is empty, we’ll prove that for all \( n \in \mathbb{N} \) we have \( n \not\in A \).

Let \( n \in \mathbb{N} \), and assume that for all \( k \in \mathbb{N} \), if \( k < n \) then \( k \not\in A \). Using the technique of strong induction, all we have to do is show that \( n \not\in A \). We argue by contradiction: suppose \( n \in A \). Then for all \( k \in A \) we have \( k \geq n \) (by the contrapositive of our hypothesis). But then \( n \) is the smallest element of \( A \), which is a contradiction. Therefore \( n \not\in A \). □

I also wanted to make a couple of comments on the last theorem we did in class on Section 6.4: irrationality of \( \sqrt{2} \). First of all, the way we used the Well-Ordering Principle to get our contradiction was really just an elementary way to say that we can reduce the fraction \( k/n \) so that 2 is not a common factor in the numerator \( k \) and the denominator \( n \).

The other comment is that this is perhaps the first proof by contraction, and almost certainly the first irrational number, known to humans: half a millenium BC, the Pythagoreans (yes, *that* Pythagoras) studied mathematics, philosophy, and many other things, and one of their beliefs was, in modern language, that all numbers are rational. But of course they knew the Pythagorean Theorem, so they know about \( \sqrt{2} \) (the hypotenuse of a right triangle with perpendicular sides both of length 1). The (apocryphal) story goes that, once when the Pythagoreans were on a boat voyage, one of the members of the group discovered the proof that \( \sqrt{2} \) is irrational, and rushed on deck to show the rest of the gang, whereupon they threw him overboard to cover up the discovery.