The textbook says this section is a “research project”, and says you should “try” to answer the exercises. However, we’ll treat this just like any other section, and you must do the exercises as usual.

Here are the important concepts:

**Definition.** Let \( f: A \to B \) and \( C \subseteq A \). The **image** of \( C \) under \( f \) is the set \( f(C) \subseteq B \) defined by

\[
f(C) = \{ f(x) \mid x \in C \}.
\]

Note that the image \( f(C) \) is described above as an “indexed family” according to the textbook’s terminology in Section 2.3 (I would use the term “indexed set”), with the index set being \( C \). As we saw in Section 2.3, this indexed set can also be described as:

\[
f(C) = \{ y \in B \mid \text{there exists } x \in C \text{ such that } y = f(x) \}.
\]

Actually, in this latter description, the requirement “\( y \in B \)” is unnecessary, because the later condition that \( y \) be of the form \( f(x) \) for some \( x \in C \) certainly puts \( y \) in \( B \), since every value of \( f \) is in \( B \). So, we could alternatively write

\[
f(C) = \{ y \mid \text{there exists } x \in C \text{ such that } y = f(x) \}.
\]

But in fact it’s more customary to write it with the “\( y \in B \)”.

**Definition.** Let \( f: A \to B \) and \( D \subseteq B \). The **inverse image** of \( D \) under \( f \) is the set \( f^{-1}(D) \subseteq A \) defined by

\[
f^{-1}(D) = \{ x \in A \mid f(x) \in D \}.
\]

**Warning.** The notation for images is unfortunate, because it looks like the notation for values: if \( f: A \to B \) and \( x \in A \), recall that the value of \( f \) at \( x \) is written \( f(x) \). To make this even worse, this value \( f(x) \) is also called the “image of \( x \) under \( f \)” — this is the same terminology we are now using for the image of a subset of the domain! Thus, you have to be very careful when using the notation for images, so that from the context it should be clear whether you mean the image of a subset of the domain or the image of an element of the domain. Believe it or not, with a little practice this doesn’t cause confusion.

The notation for inverse images is **even more dangerous**, because it looks like we’re using an inverse function. But we’re not! Recall that if function \( f: A \to B \) then the inverse relation \( f^{-1} \) from \( B \) to \( A \) is in
fact a function from \( B \) to \( A \) if and only if \( f \) is 1-1 onto. But we don’t need any condition on a function \( f: A \to B \) in order to form inverse images \( f^{-1}(D) \) of subsets \( D \) of \( B \). Fortunately, at least the two uses of the notation \( f^{-1} \) don’t lead to any contradictions, because if in fact \( f \) is 1-1 onto, so that the inverse relation is a function \( f^{-1}: B \to A \), then the inverse image of a subset \( D \subseteq B \) under \( f \) is the same as the image of \( D \) under the function \( f^{-1} \). Again, you must be very careful when using the notation for inverse images.