Name:

- No books, notes or calculators of any kind are permitted on the test.
- Do not use your own scratch paper.
- Write each solution in the space provided, not on scratch paper.
- If you need more room, write on the back of the page. If you still need more room, ask for scratch paper.
- Be sure to give reasons for your answers (except where explicitly told otherwise)!
- Your solutions must be complete and organized, otherwise points may be deducted.
- There are 5 problems, and all problems have equal credit.
1. The parts of this problem are independent.
In this problem, only the answer will be graded; you do not need to show any work!

(a) If the characteristic polynomial of $A$ is $(1 - \lambda)^2(5 - \lambda)$, what are the possible values of the dimension of the eigenspace associated to the eigenvalue 1?

Solution: 1, 2

(b) What are the eigenvalues of the matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
4 & 5 & 6
\end{bmatrix}
\]?

Solution: 1, 3, 6

(c) If the determinant of
\[
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
l & m & n & o \\
p & q & r & s
\end{bmatrix}
\] is 7, what is the determinant of
\[
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
3l & 3m & 3n & 3o \\
p & q & r & s
\end{bmatrix}
\]?

Solution: $3(7) = 21$

(d) If the determinant of
\[
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
l & m & n & o \\
p & q & r & s
\end{bmatrix}
\] is 7, what is the determinant of
\[
\begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
l & m & n & o \\
p & q & r & s
\end{bmatrix}
\]?

Solution: $-7$

(e) If the determinant of
\[
\begin{bmatrix}
a & b & c \\
de & e & f \\
g & h & i
\end{bmatrix}
\] is 5 and
\[
\begin{bmatrix}
a & b & c \\
de & e & f \\
g & h & i
\end{bmatrix} \overset{3R_1 + R_2}{\longrightarrow} \begin{bmatrix}
a & b & c \\
0 & j & k \\
g & h & i
\end{bmatrix},
\] what is the determinant of
\[
\begin{bmatrix}
a & b & c \\
0 & j & k \\
g & h & i
\end{bmatrix}
\]?

Solution: 5

(f) $A$ is a 4 \times 4 matrix with eigenvalues $\lambda_1$ and $\lambda_2$. The eigenspace associated with $\lambda_1$ has the basis \{v_1\}. The eigenspace associated with $\lambda_2$ has the basis \{v_2, v_3\}. Is $A$ diagonalizable?

Solution: No.
2. Apply the Gram-Schmidt algorithm to transform the given basis \( \{v_i\} \) into an orthogonal basis \( \{u_i\} \):

\[
v_1 = (1, -1, 1, -1), \quad v_2 = (1, 3, 1, -1), \quad v_3 = (2, 0, 1, 1)
\]

**Solution:** We take

\[
u_1 = v_1 = (1, -1, 1, -1)
\]

Next,

\[
u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = v_2 - 0u_1 = v_2 = (1, 3, 1, -1)
\]

Then

\[
u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = v_3 - \frac{2}{12} u_1 - \frac{2}{12} u_2 = v_3 - \frac{1}{6} u_1 - \frac{1}{6} u_2
\]

To avoid fractions, we multiply by 6 and take

\[
u_3 = 6v_3 - 3u_1 - u_2 = (12, 0, 6, 6) - (3, -3, 3, -3) - (1, 3, 1, -1) = (8, 0, 2, 10)
\]

Thus the orthogonal basis is

\[
u_1 = (1, -1, 1, -1), \quad u_2 = (1, 3, 1, -1), \quad u_3 = (8, 0, 2, 10)
\]
3. The parts of this problem are independent.

(a) Which one is the characteristic polynomial of the matrix \[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
\end{pmatrix}
\]?

A: \(-2 - 5\lambda + \lambda^2\) \hspace{1cm} B: \(-4 + 3\lambda + \lambda^2\)

**Solution:** A

computation:

\[
\begin{vmatrix}
1 - \lambda & 2 \\
3 & 4 - \lambda \\
\end{vmatrix} = (1 - \lambda)(4 - \lambda) - 6 = 4 - 5\lambda + \lambda^2 - 6
\]

(b) If the characteristic polynomial of a matrix is \(6 - 5\lambda + \lambda^2\), what are the eigenvalues?

**Solution:** 2, 3

factored: \((2 - \lambda)(3 - \lambda)\)

(c) Given that \(\lambda = 2\) is an eigenvalue of the matrix \(A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}\), find a basis of the associated eigenspace.

**Solution:** \((2, 1)\)

because

\[
A - 2I = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \text{REF:} \quad \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}
\]
4. The parts of this problem are independent.

(a) Given that $A$ and $B$ are $3 \times 3$ matrices with $\det A = 3$ and $\det B = 5$, find: $\det(2A)$

Solution: $2^3 \det(A) = 24$

(b) Given that $A$ and $B$ are $3 \times 3$ matrices with $\det A = 3$ and $\det B = 5$, find: $\det(AB^{-1})$

Solution: $\det(A)\det(B^{-1}) = \det(A)\frac{1}{\det(B)} = \frac{3}{5}$

(c) Evaluate the determinant by inspection. Give reasons for your answer.

\[
\begin{bmatrix}
3 & 0 & -1 & 0 \\
7 & 2 & -3 & 0 \\
8 & 6 & 4 & 0 \\
4 & 9 & 2 & 0 \\
\end{bmatrix}
\]

Solution: 0 because there is a column of zeros
5. Given that the $5 \times 5$ diagonalizable matrix $A$ has

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>basis for eigenspace</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>(-1,0,1,2,3)</td>
</tr>
<tr>
<td>2</td>
<td>(0,1,-1,3,0), (1,-1,0,1,1)</td>
</tr>
<tr>
<td>3</td>
<td>(2,1,2,0,0), (-3,0,0,-1,2)</td>
</tr>
</tbody>
</table>

Find a diagonalizing matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

**Solution:** Use the eigenvectors as the columns of $P$:

$$P = \begin{bmatrix}
-1 & 0 & 1 & 2 & -3 \\
0 & 1 & -1 & 1 & 0 \\
1 & -1 & 0 & 2 & 0 \\
2 & 3 & 1 & 0 & -1 \\
3 & 0 & 1 & 0 & 2
\end{bmatrix}$$

Then for $D$ use the associated eigenvalues along the diagonal in the same order:

$$D = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix}$$