Name:

- No books, notes or calculators of any kind are permitted on the test.
- Do not use your own scratch paper.
- Write each solution in the space provided, not on scratch paper.
- If you need more room, write on the back of the page. If you still need more room, ask for scratch paper.
- Be sure to give reasons for your answers (except where explicitly told otherwise)!
- Your solutions must be complete and organized, otherwise points may be deducted.
- There are 5 problems, and all problems have equal credit.
1. The parts of this problem are independent.
In this problem, only the answer will be graded; you do **not** need to show any work!

(a) If $A$ is a $3 \times 5$ matrix, what are the possible values of the rank of $A$?

Solution: 0, 1, 2, 3

(b) Given that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

do the columns of $A$ span $\mathbb{R}^3$?

A: yes     B: no

(c) Given the vectors $v_1 = (1, 2, 0, -1)$, $v_2 = (2, 1, 3, 1)$, $v_3 = (4, 5, 3, -1)$, we have a standard method, using a certain matrix $A$, to find a subset of $\{v_1, v_2, v_3\}$ which is a basis for $\text{span}\{v_1, v_2, v_3\}$. What is the matrix $A$ in this case?

(d) Do the vectors $v_1 = (1, 2)$ and $v_2(0, 3)$ form a basis for $\mathbb{R}^2$?

A: yes     B: no

(e) If $A$ is a $2 \times 7$ matrix with rank 2, what is the dimension of the null space of $A$?

Solution: 5

(reason: $\text{rank}(A) + \text{nullity}(A) = n$ if $A$ is $m \times n$)

(f) If the columns of $A$ are independent, how many solutions does the homogeneous system $Ax = 0$ have?

A: one     B: none     C: infinitely many
2. The parts of this problem are independent.

(a) Let \( W \) be the set of all vectors \((x_1, x_2)\) in \( \mathbb{R}^2 \) such that \( x_1 \geq 0 \). Is \( W \) a subspace of \( \mathbb{R}^2 \)? Why or why not?

\[
\text{Solution: No, because, for example, the vector } (1, 0) \text{ is in } W \text{ but } -2(1, 0) = (-2, 0) \text{ is not in } W, \text{ so } W \text{ is not closed under scalar multiplication.}
\]

(b) If \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{b} \) are vectors in \( \mathbb{R}^5 \) such that the reduced echelon form of the matrix
\[
\begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
express \( \mathbf{b} \) as a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3. \)
3. Given that \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \) has reduced echelon form \( \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \), find:

(a) a basis for the column space of \( A \)

(b) a basis for Row \( A \)
4. The parts of this problem are independent.

(a) If

\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\]

find the augmented matrix of the normal system associated to the system \( Ax = b \).

(b) Given that \( \tilde{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) is a least squares solution of the system

\[
\begin{align*}
2x_1 - x_2 &= 1 \\
x_1 + x_2 &= 1 \\
x_1 - x_2 &= 5
\end{align*}
\]

find the orthogonal projection of the vector \((1, 1, 5)\) on the subspace spanned by \((2, 1, 1)\) and \((-1, 1, -1)\).
5. Let $V = \text{span}\{(3, 6, -1, -6), (2, 4, 1, -4), (3, 6, 1, -4)\}$. Given that $A = \begin{bmatrix} 3 & 6 & -1 & -6 \\ 2 & 4 & 1 & -4 \\ 3 & 6 & 1 & -6 \end{bmatrix}$ has reduced echelon form $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find:

(a) a basis for the orthogonal complement $V^\perp$

(b) the dimension of $V$