• Be sure to show all your work!
• Be sure to give reasons for your answers!
• Points will probably be taken off if you do not give a clear indication of the method and write your solution neatly, in order, and clearly indicate your final answer.
• Points may be taken off for a correct answer with a mistake in the supporting work.
• No books, notes or calculators of any kind are permitted on the test.
• Write each solution in the space provided, not on scratch paper.
• If you need more room, write on the back of the page.
• Time limit: 1.5 hours (90 minutes). Points may be deducted for exceeding this limit.
1. **In this problem only**, just circle the correct answer — you do not need to give reasons, and only the answer will be graded:

(a) If the augmented matrix of a system is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

how many solutions does the system have?

A: one  B: none  C: infinitely many

(b) If \( A \) is \( 5 \times 8 \) and \( B \) is \( 3 \times 5 \), then \( B^T A \) exists.

True  False

(c) For any \( 4 \times 3 \) matrix \( A \) and any \( b \in \mathbb{R}^4 \), the system \( A^T Ax = A^T b \) is consistent.

True  False

(d) If \( A \) is an invertible matrix with all integer entries, and \( \det(A) = 1 \), then \( A^{-1} \) also has all integer entries.

True  False

(e) It is impossible for 0 to be an eigenvalue of a matrix.

True  False

(f) The vectors \((1, -1, 1, 1)\) and \((1, 2, 1, -1)\) in \( \mathbb{R}^4 \) are orthogonal.

True  False
2. In each part of this problem, you are given a transformation $T$ from one Euclidean space to another. Your job is to determine whether $T$ is linear.

(a) $T(x, y) = (x^2, y)$

(b) $T(x, y) = (2x + y, x)$
3. Suppose $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear and $T(v_i) = w_i$ for $i = 1, 2$, where

\[
\begin{align*}
  v_1 &= (1, 2) & v_2 &= (1, 3) \\
  w_1 &= (1, 1, 0) & w_2 &= (2, -1, 1).
\end{align*}
\]

Find the matrix $A$ such that $T(x) = Ax$. 
4. Consider the linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^4$ such that $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 2 & 1 & -2 & 1 \\ -1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 2 & -6 & 3 \\ 2 & 4 & 1 & -1 & 5 \end{bmatrix}$.

Given that reduced echelon form of $A$ is

$$
\begin{bmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

(a) Find a basis for the range of $T$.

(b) Find a basis for the kernel of $T$. 
5. **Use elementary row operations** to find the inverse of the matrix \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \). Be sure to put the final answer in the indicated location!

\[
A^{-1} = \]

6. Let \( A = \begin{bmatrix} 2 & 5 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & -7 \end{bmatrix} \).

(a) What are the eigenvalues of \( A \)?

(b) Is \( A \) diagonalizable? (Just circle the correct answer.)

   yes   no

(c) If your answer above was “yes”, then write a diagonal matrix \( D \) similar to \( A \), while if your answer was “no” then give a brief reason.
7. (a) Use the Gram-Schmidt Algorithm to convert the given basis \( \{ \mathbf{v}_i \} \) into an orthogonal basis \( \{ \mathbf{u}_i \} \). Be sure to write the final form of your vectors \( \mathbf{u}_1, \mathbf{u}_2 \) in the indicated locations!

\[
\mathbf{v}_1 = (1,1,1,1) \quad \mathbf{v}_2 = (1,1,1,0)
\]

\[
\mathbf{u}_1 = \quad \mathbf{u}_2 =
\]

(b) If \( A \) is a \( 4 \times 7 \) matrix with rank 4, then there exists a vector \( \mathbf{b} \) such that the system \( Ax = \mathbf{b} \) is inconsistent. Give a brief reason.

True \quad False

(c) If \( A \) and \( B \) are \( n \times n \) matrices such that \( AB \) is invertible, then \( A \) is invertible. Give a brief reason.

True \quad False