The Cuntz-Pimsner Inevitability

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Directed graphs

\[ E = (E^0, E^1, r, s) \quad r = \text{range}, \ s = \text{source} \]

edges \[ E^1 \xrightarrow{r} E^0 \]

vertices

\[ \bullet v \leftarrow \underbrace{e} \bullet u \]
Representations

\[(S, P) : E \to B \quad E^1 \xrightarrow{S} B \xleftarrow{P} E^0 \quad \text{such that:}\]

- \(S_e\) partial isometries
- \(P_v\) mutually orthogonal projections
- \(S_e^*S_e = P_{s(e)}\)
- \(P_v = \sum_{r(e) = v} S_eS_e^*\) if \(0 < |r^{-1}(v)| < \infty\)
- \(\sum_{e \in F} S_eS_e^* \leq P_v\) for any finite nonempty subset \(F \subseteq |r^{-1}(v)|\)
universal representation \((s, p) : E \to C^*(E)\), i.e., for every representation \((S, P) : E \to B\),

\[
\begin{array}{ccc}
E & \xrightarrow{(s,p)} & C^*(E) \\
\downarrow & & \downarrow \\
(S,P) & \downarrow & B \\
& & \downarrow \times P \\
& & S \times P
\end{array}
\]
Examples

\[ \begin{align*}
E: & \quad \bullet \; \nu \\
C^*(E) & = \mathbb{C} \\
generator: & \quad p_\nu = 1
\end{align*} \]
Examples

\[ E: \quad u \xleftarrow{e} v \]

\[ C^*(E) = M_2 \]

generators: \( s_e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( p_u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( p_v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)
Examples

\[ E: \]

\[ C^*(E) = M_3 \]

generators:

\[
\begin{align*}
  s_e &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
  s_f &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
  p_u &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
  p_w &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
  p_v &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]
Examples

$E: \quad \nu \overset{\sim}{\rightarrow} e$

$C^*(E) = C(\mathbb{T})$

generators: $s_e = z, \ p_\nu = 1$

universal $C^*$-algebra generated by a unitary
Examples

\[ E: \quad f \xhookleftarrow{\mapsto} v \xhookrightarrow{\mapsto} e \]

\[ C^*(E) = \mathcal{O}_2 \] — a very interesting \( C^* \)-algebra!

generators:
\( s_e = \) isometry onto one half of Hilbert space
\( s_f = \) isometry onto other half of Hilbert space
\( p_v = 1 \)

Any nonzero representation will do, because \( \mathcal{O}_2 \) is simple.
Let \((S, P) : E \to B\) be rep with \(P_v \neq 0\) for all \(v\). Then:

1. \(\{S(E^1), P(E^0)\}\) is faithful image of \(E\) in \(B\).

2. \(\{S_e : e \in E^1\}\) and \(\{P_v : v \in E^0\}\) are both linearly independent.

3. \(P_u S_e P_v = S_e\) if \(u = r(e)\) and \(v = s(e)\), and =0 else.

4. \(S_e^* S_f \neq 0\) implies \(e \neq f\).

5. \(S_e S_f^* \neq 0\) implies \(s(e) = s(f)\).
Consider linear spans

Define

- $\mathcal{X} = \text{span}\{S_e : e \in E^1\}$, a subspace of $B$
- $\mathcal{A} = \text{span}\{P_v : v \in E^0\}$, a commutative $C^*$-subalgebra of $B$

Write

- $x \in \mathcal{X}$ as $x = \sum_e x(e)S_e$
- $a \in \mathcal{A}$ as $a = \sum_v a(v)P_v$ (Note: $a \in c_0(E^0)$.)

Pull back to $E$:

- $X_0$ = vector space with basis $\{\chi_e : e \in E^1\}$
- $A = c_0(E^0)$

Write

- $x \in X_0$ as $x = \sum_e x(e)\chi_e$
- $a \in A$ as $a = \sum_v a(v)\delta_v$
Properties yield more properties

The properties of \((S, P)\) give:

\[
\left( \sum_{v} a(v) P_v \right) \left( \sum_{e} x(e) S_e \right) = \sum_{e} a(r(e)) x(e) S_e
\]

\[
\left( \sum_{e} x(e) S_e \right) \left( \sum_{v} a(v) P_v \right) = \sum_{e} x(e) a(s(e)) S_e
\]

These lead us to define an \(A\)-bimodule structure on \(X_0\):

\[
(a \cdot x)(e) = a(r(e)) x(e)
\]

\[
(x \cdot a)(e) = x(e) a(s(e))
\]
The property that $S_e^* S_f$ is either the source projection of $S_e$ if $e = f$ or 0 if $e \neq f$ reminds us of an orthonormal set. Extending to linear combinations:

\[
(\sum_e x(e) S_e)^* (\sum_f y(f) S_f) = \sum_v \left( \sum_{s(e) = v} \overline{x(e)} y(e) \right) P_v
\]

This leads us to define an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $X_0$ by

\[
\langle x, y \rangle(v) = \sum_{s(e) = v} \overline{x(e)} y(e)
\]

Note: this is linear in the 2nd variable instead of the first, so we’re using a “physicist’s inner product”!
Correspondences

It takes some effort, but this has all the properties we expect of an inner product:

- $\langle x, x \rangle \geq 0$, and $=0$ iff $x = 0$
- $\langle x, y \rangle^* = \langle y, x \rangle$
- $\langle x, y \cdot a \rangle = \langle x, y \rangle a$
- $\|x\| = \sqrt{\|\langle x, x \rangle\|}$ is a norm.

Let $X$ be the completion. Then the bimodule operations and the inner product extend, with the same properties, and we call $X$ an $A$-correspondence. If we forget the left $A$-module structure it’s called a Hilbert $A$-module — like a Hilbert space, but $A$ plays the role of the scalars.
There’s a subtlety: we expect that if $T$ is a bounded linear operator on $X$ then $T$ has an adjoint $T^*$:

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for all } x, y \in X.$$  \hspace{1cm} (1)

Sadly (or interestingly) this turns out to not be automatic, so we must explicitly require our operators to have adjoints, and then the set $\mathcal{L}(X)$ of all adjointable operators becomes a $C^*$-algebra similar to $B(H)$ for a Hilbert space $H$.

Happily, a quick computation with the generators shows that the left-module operators $\phi(a)x = a \cdot x$ are adjointable:

$$\langle a \cdot x, y \rangle = \langle x, a^* \cdot y \rangle$$
Our experience with Hilbert space leads us to define, for $x, y \in X$, a rank one operator $\theta_{x,y}$ on $X$ by

$$\theta_{x,y}z = x \cdot \langle y, z \rangle.$$ 

This is adjointable, with $\theta_{x,y}^* = \theta_{y,x}$. The closed span of the rank ones is the $C^*$-algebra $\mathcal{K}(X)$ of compact operators.

Well, not compact in the classical sense...
Connect \((S, P)\) with \(X\)

There are a unique bounded linear map \(\psi : X \to B\) and a unique homomorphism \(\pi : A \to B\) such that

- \(\psi(\chi_e) = S_e\)
- \(\pi(\delta_v) = P_v\)

Computations with the generators show:

- \(\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b)\)
- \(\pi(\langle x, y \rangle) = \psi(x)*\psi(y)\)
It takes a bit of work to see that there is a unique homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that

$$\psi^{(1)}(\theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for all } x, y \in X.$$
What about the **Cuntz-Krieger property**

\[ P_v = \sum_{r(e)=v} S_e S_e^* \quad \text{if} \quad 0 < |r^{-1}(v)| < \infty ? \]
A moment’s thought reveals that $\phi(\delta_v)$ is the unique bounded operator on $X$ such that

$$\phi(\delta_v)\chi_e = \begin{cases} \chi_e & \text{if } r(e) = v \\ 0 & \text{if not} \end{cases}$$

so if $0 < |r^{-1}(v)| < \infty$ this quickly gives

$$\phi(\delta_v) = \sum_{r(e) = v} \theta_{\chi_e, \chi_e}$$
and hence

\[ \pi(\delta_v) = P_v \]
\[ = \sum_{r(e)=v} S_e S_e^* \]
\[ = \sum_{r(e)=v} \psi(\chi_e)\psi(\chi_e)^* \]
\[ = \sum_{r(e)=v} \psi^{(1)}(\theta \chi_e, \chi_e) \]
\[ = \psi^{(1)} \left( \sum_{r(e)=v} \theta \chi_e, \chi_e \right) \]
\[ = \psi^{(1)}(\phi(\delta_v)) \]
It’s not hard to check that
- $\phi(\delta_v) \in \mathcal{K}(X)$ if and only if $|r^{-1}(v)| < \infty$
- $\delta_v \ker \phi = \{0\}$ if and only if $0 < |r^{-1}(v)|$.

It now follows that by linearity and density, for all $a \in A$ such that $\phi(a) \in \mathcal{K}(X)$ and $a \ker \phi = \{0\}$ we have

$$\pi(a) = \psi^{(1)} \circ \phi(a).$$

Such $a$’s form the **Katsura ideal** $J_X$.

We’re concluding that the Cuntz-Krieger property of $(S, P)$ is equivalent to the following:

$$\pi(a) = \psi^{(1)} \circ \phi(a) \text{ for all } a \in J_X.$$
Pair \((\psi, \pi)\) consisting of a linear map \(\psi : X \rightarrow B\) and a homomorphism \(\pi : A \rightarrow B\) such that

- \(\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b)\)
- \(\pi(\langle x, y \rangle) = \psi(x)^*\psi(y)\)
- \(\pi(a) = \psi^{(1)} \circ \phi(a)\) for all \(a \in J_{\mathcal{X}}\).
universal representation \((k_X, k_A) : (X, A) \to \mathcal{O}_X\), i.e., for every representation \((\psi, \pi) : (X, A) \to B\),
The assignments \((S, P) \mapsto (\psi, \pi)\) give a bijection between representations of \(E\) and of \(X\), and this gives an isomorphism

\[ C^*(E) \simeq \mathcal{O}_X. \]
The theory of $C^*$-correspondences and their Cuntz-Pimsner algebras has one more nugget for us. We’ve discussed how every graph gives rise to a correspondence. There’s a satisfying converse:

**Theorem (Kaliszewski-Patani-Q)**

*If $A$ is commutative with discrete spectrum $\hat{A}$ and $X$ is an $A$-correspondence, then there is a directed graph $E$, unique up to isomorphism, such that $E^0 = \hat{A}$ and $X$ is isomorphic to the associated $c_0(E^0)$-correspondence.*

(fine print: require $X$ nondegenerate),