Modeling of Supply Chains and Production Systems

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Prototypical Topologies

(a) i) linear production line, ii) re-entrant production line

(b) iii) supply chain, iv) reverse arrows: distribution network
The air traffic network (courtesy of Dirk Brockmann and "Where is George")
## Process characteristics I

### Machine process
- machine failure is stochastic
- known MTF and MTR, usually no probability distribution
- large stochastic influence on flow - machine is on or off.
- small variation in the actual processing time, usually a probability distribution

### Transportation process
- delay along a link might be large
- variation in the actual transportation time, usually a probability distribution
Human interaction

- typically characterizes the fill or taking of parts from buffer
- also associated with transport
- typically small variation in processing time
What do we want to know about the system?

Questions

1. If an item is started at $t_0$ when does it come out? $\rightarrow p(\tau)$.
2. If every node is characterized by a capacity, and every item has a schedule, i.e. a path through the network, what is the capacity of the system, i.e.

$$C = \text{flow} = \frac{\# \text{ of products}}{\text{time}}$$

specifically its
- mean
- instantaneous capacity
- time evolution
Question

Given a demand rate $d(t), t \in [0, T]$. Given the state of the factory i.e. the position of the work in progress (WIP) and the state of the machines.

What is the start pattern $\lambda(t), t \in [0, T]$, s.t, the output of the factory is optimally close to the target demand
Multiclass networks

Assume

- several classes of items, e.g. red, green, blue.
- each one has a particular processing rate, $\mu_r, \mu_g, \mu_b$.
- demand signals $d_r(t), d_g(t), d_b(t)$
- WIP distribution over the factory

Dispatch policy

Choose the next item to work on in a way that minimizes the objective

$$P = c_r \|\text{out}_r(t) - d_r(t)\|^2 + c_g \|\text{out}_g(t) - d_g(t)\|^2 + c_b \|\text{out}_b(t) - d_r(t)\|^2$$
### Discrete event simulations

- **State:**
  - Position of every item at every time
  - Status of machine, working, down, available

- **Event list:**
  - Time when machine M1 is finished
  - Time when machine M3 is repaired
  - Time when part $x$ is released into factory
  - Etc.

- At every event there is a state transition and the event list will be updated.

Software available, e.g. $\chi$ from TU Eindhoven
### Queueing Networks

- The buffer - machine unit is called queue.
  - arrival process with rate $\lambda(t)$
  - departure process with rate $\mu(t)$

- Simplest cases:
  - M/M/1 queue
  - M/M/n queue
  - M/M/1/K queue
  - etc
### Steady State for M/M/1 queue

- **mean cycle time:**
  \[ \tau = \frac{1}{\mu - \lambda} \]

- **mean queue length:**
  \[ q = \frac{\lambda}{\mu - \lambda} \]

- Little’s law - true for any reasonable stochastic process (not only M/M/1) since it is just mass conservation:
  \[ \lambda = \frac{q}{\tau} \]
Most important concept characterizing the steady state behavior of a queue

\[ F = \lambda = \frac{\mu}{1 + q} \]
ODE models

Define \( x_i(t) \) to be a continuous variable describing the length of the \( i_{th} \) queue as a function of time.

\[
\frac{dx_i}{dt}(t) = \lambda_{in} - \mu_{out}
\]

in general

\[
\mu_{out} = \begin{cases} 
\mu_i & \text{if } x_i > 0 \\
\lambda_{in} & \text{if } x_i = 0 
\end{cases}
\]

This leads to non-smooth dynamical systems. Issues are:

- stability
- control
- dynamics - periodic orbits, chaos etc.
### Equivalence between the fluid model and the queuing system

We would expect that the time evolution of $x(t)$ in some sense reflects the long term behavior of the queuing system. The following is not a theorem:

**Theorem**

*A multiclass queueing network is stable iff the fluid network is stable*

There are partial results that show that you need additional constraints on the dispatch rules.
Fundamental Idea:

Notice that fluid models are not a fluid - they lead to ODEs. Continuum in production but not in production stages. Hence: **Model high volume, many stages, production via a fluid.**

Basic variable

- product density (mass density) $\rho(x, t)$.
- $x$ is the position in the production process, $x \in [0, 1]$.
- degree of completion
- stage of production
Mass conservation and state equations

Mass conservation and state equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0
\]

\[F = \rho v_{eq}\]

Typical models for the equilibrium velocity $v_{eq}$ (state equation) are

\[v_{eq}^{traffic}(\rho) = v_0 (1 - \frac{\rho}{\rho_c})\]

\[v_{eq}^Q = \frac{\mu}{1 + q}\]

\[v_{eq} = \Phi(q)\]

with $q$ the total load (WIP) given as $q(\rho) = \int_0^1 \rho(x, t)dx$
Note

- Model is an adiabatic model, i.e. velocity (flux) changes instantaneously with a change in total load
- \( \Phi(q) \) may be determined experimentally or theoretically
- Clearing function is \( F = qv_{eq}(q) = q\Phi(q) \).

Clearing function again
Validation: DES vs. fluid simulation

Figure: Throughput as a function of time for a sinusoidally varying input
Moment expansions


Derives the transport equation from a probability distribution for the speed of the part through the factory. Extends the adiabatic model to an equation for the time evolution of the velocity which happens to be Burgers equation.
Advection-Diffusion equation

D. Armbruster, C. Ringhofer,

Derives

\[
\frac{\partial \rho}{\partial t} + v_{eq} \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2} \tag{1}
\]

\[v_{eq}(t) = \Phi(q(\rho(t))) \tag{2}\]

via a Chapman-Enskog expansion.
Control via influx

Michael La Marca, Dieter Armbruster, Michael Herty and Christian Ringhofer
Controls the outflux of a re-entrant factory via its influx. We define a cost functional $J$ by

$$J(\rho, \lambda) := \frac{1}{2} \int (d(t) - v(\rho)\rho(1, t))^2 \, dt.$$  \hspace{1cm} (3)

and optimize

$$\min J(\rho, \lambda) \text{ subject to}$$ \hspace{1cm} (4a)

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left( v(\rho)\rho(x, t) \right) = 0$$ \hspace{1cm} (4b)

$$\lambda(t) = v(\rho)\rho(0, t)$$ \hspace{1cm} (4c)

$$\rho_0(x) = \rho(x, 0)$$ \hspace{1cm} (4d)
**Figure:** Sinusoidal target outflux (green), needs sawtooth influx (red) to generate matching outflux (blue)
Data analysis

In general there are not a whole lot of inverse problems in production systems. I can see one: Data analysis of sanitized data of a real INTEL factory for about 3 months production.

**Details:**

- 920 lots
- time log in and out at all machines
- identify 8 approximately equally spaced machines
- Determine time-in at all 8 machines
Use these data to find the best fit for the clearing function and the diffusion coefficient (possibly as a function of position). This has been Ali Unver’s thesis but could be revisited for this purpose.
An optimization problem

Replace Lamarca’s cost functional

\[ J(\rho, \lambda) := \frac{1}{2} \int (d(t) - v(\rho)\rho(1, t))^2 \, dt. \] (5)

by

\[ J(\rho, \lambda) := c_1 \int (d(t) - v(\rho)\rho(1, t))H(d(t) - v(\rho)\rho(1, t))dt + c_2 \int (v(\rho)\rho(1, t) - d(t))H(v(\rho)\rho(1, t) - d(t))dt. \] (6)

\[ c_2 \int (v(\rho)\rho(1, t) - d(t))H(v(\rho)\rho(1, t) - d(t))dt. \] (7)

where storage cost \( c_2 \) are much smaller than penalty costs \( c_1 \).
Aggregation

Very important problem in network flows:
Characterize a subnetwork as a single node
Characterize the clearing function of the aggregate node as a function of the parameters of the nodes of the subnetwork. Can be done for a balanced Jackson network.

Result:

\[ F = \frac{\mu}{n + q} \]

where \( n \) is the number of nodes, independent of the topology of the network, \( q \) is the total number of jobs waiting in the subnetwork.

Not many networks are Jackson - can one do some perturbation theory of almost Jackson networks?
Dynamics

The event: There is a blizzard in the midwest that shuts down O’Hare and all airports within a 50 mile radius.

- All traffic originating in the shut airports and all traffic ending in these airports will be cancelled.
- Traffic going from place A to place B through an airport that is closed, will try to re-route.
- A network where links represent open seats will collapse over time.

Can we predict and control the total wait time of all affected passengers as a function of the network traffic load at the time of the event?