Brain Surface Conformal Spherical Mapping

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Abstract

It is well known and proved that any genus zero surface can be conformally mapped onto the sphere. In this paper, we introduce a composite map which finds a conformal mapping between brain surface and a sphere based on the paper Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping (X.Gu et al. 2002) [1]. We also conduct experiment based on the algorithm provided by X.Gu (2002), showing the availability for Brain Surface Conformal Spherical Mapping.

1. Introduction

Any genus zero surface is homeomorphic to a sphere. Moreover, any genus zero surface is conformally homeomorphic to a sphere which preserves angles. Since the cortical surface of the brain is a genus zero surface, conformal mapping offers a convenient method to retain local geometric information, when mapping data between surfaces. In this paper, we introduce a genus zero surface conformal mapping algorithm and demonstrate its use in computing conformal mappings between brain surfaces. The algorithm is based on the theorem that:

**Theorem 1.** Maps of genus zero Riemannian Surfaces are conformal if and only if they are harmonic.

The proof can be found in [2].

The algorithm is trying to minimize a harmonic energy function to get the harmonic map between brain surface and a brain sphere such that they are conformal. The algorithm depends only on the surface geometry and is invariant to changes in image resolution and the specifics of the data triangulation. Also, an experiment is conducted showing that the algorithm has advantageous properties for cortical surface matching.

This paper is organized as followings: section 2 will talk about some terms and definitions; section 3 will discuss the triangulation algorithm Marching Cubes Algorithm.
Section 4 will discuss the conformal mapping algorithm while section conducts the experiment based on the algorithm.

2. Mesh, Gauss Map and Energy Functions

**Definition 1.** Suppose K is a simplicial complex, and \( f: |K| \to R^3 \), which embeds \(|K|\) in \(R^3\); then \((K,f)\) is called a mesh

**Definition 2.** All piecewise linear functions defined on \(K\) form a linear space, denoted by \(C_{PL}(K)\).

**Definition 3.** Suppose \(M_1\) is a mesh, a Gauss map \(N: M_1 \to S^2\) is defined as:
\[
N(v) = \vec{n}(v), v \in M_1
\]
\(\vec{n}(v)\) is the normal at \(v\)

**Definition 4.** Suppose \(f \in C_{PL}(K)\), \(\{u, v\}\) is an edge, the Tuette energy is defined as:
\[
E(f) = \langle f, f \rangle = \sum_{\{u,v\} \in K} ||f(u) - f(v)||^2
\]

**Definition 5.** Suppose edge \(\{u, v\}\) has two adjacent faces \(T_\alpha, T_\beta\), for face \(T_\alpha = \{v_0, v_1, v_2\}\), define the parameters:
\[
\begin{align*}
\alpha_{v_1,v_2}^\alpha &= \frac{1}{2} \frac{(v_1 - v_3) \cdot (v_2 - v_3)}{(v_1 - v_3) \times (v_2 - v_3)} \\
\alpha_{v_2,v_3}^\alpha &= \frac{1}{2} \frac{(v_2 - v_1) \cdot (v_3 - v_1)}{(v_2 - v_1) \times (v_3 - v_1)} \\
\alpha_{v_3,v_1}^\alpha &= \frac{1}{2} \frac{(v_3 - v_2) \cdot (v_3 - v_2)}{(v_3 - v_2) \times (v_3 - v_2)}
\end{align*}
\]

\(T_\beta\) is defined similarly. The harmonic energy is defined as:
\[
E(f) = \langle f, f \rangle = \sum_{\{u,v\} \in K} (\alpha_{u,v}^\alpha + \alpha_{u,v}^\beta) ||f(u) - f(v)||^2
\]

3. Marching Cubes Algorithm and 3D image Triangulation

Marching cubes is an algorithm published by Lorensen and Cline [3] for extracting a polygonal mesh of isosurface from a three dimension scalar volumetric data. Such a data set has a scalar value residing at each lattice point of a rectilinear lattice in 3D space. The
marching cubes process the volumetric data set by considering the “cubes” that make up the volume. The cubes are defined by the volume’s lattice. Each lattice point is a corner vertex of a cube. Each cube vertex that has a value equal to or above the isovalue is marked; all other vertices are left unmarked. The isosurface intersects each cube edge by one marked vertex and one unmarked vertex. Any cube that contains an intersected edge is active. The computations that find the active cubes can be viewed as the active cube determination component of the Marching Cubes. Figure 1 shows the cube and isosurface separated vertices.

![Marching Cube](image)

**Figure 1. Marching Cube**

The standard Marching Cubes algorithm extracts mesh by processing the data set in a sequential cube-by-cube manner. In the standard Marching Cubes Algorithm, there are 15 unique cube-isosurface intersection scenarios. Those 15 scenarios are stored as the lookup table for mesh extraction.

![15 Marching Cubes Scenarios](image)

**Figure 2. 15 Marching Cubes Scenarios**

With marked vertices and lookup table, we can build the triangle mesh for further research study.

### 4. Conformal Spherical Mapping
The conformal mapping \( f : M_1 \to S^2 \) can be constructed by using the steepest descent method. However, during the transformation, the major difficulty is that the solution is not unique but forms a Mobius group:

**Definition 6.** Mapping \( f : C \to C \) is a Mobius transformation if and only if

\[
f(z) = \frac{az + b}{cz + d}, a, b, c, d \in C, ad - bc \neq 0
\]

All Mobius transformations form the Mobius group. In order to determine a unique solution, we need additional constraints. In practice, we use the zero mass center constraint.

**Definition 7.** Mapping \( \vec{f} : M_1 \to M_2 \) satisfies the zeros mass-center condition if and only if

\[
\int_{M_1} \vec{f} d\sigma_{M_1} = 0,
\]

Where \( d\sigma_{M_1} \) is the unit area on \( M_1 \)

The conformal spherical mapping is a composite mapping. First we use Gauss map as the initial map, then we construct the Tuette map based on the Gauss map, finally the conformal map is built on Tuette map.

**Algorithm [1]:**

*Input (Mesh M, step length \( \delta t \), energy difference threshold \( \delta E \)), output (\( \vec{h} : M \to S^2 \)), here \( \vec{h} \) minimize the harmonic energy and satisfies the zero mass center constraint.*

1. Compute the Gauss map \( N : M \to S^2 \), compute Tuette energy \( E_0 \).
2. Use the steepest decent Algorithm to minimize the Tuette energy and update the map \( N \) iteratively. When achieves the threshold \( \delta t \), stops and let \( \vec{\nu} = N \), thus \( \vec{\nu} \) is the Tuette map.
3. Let \( \vec{h} = \vec{\nu} \), compute the Tuette energy \( E_1 \); for each vertex \( v \in M \), compute the absolute derivative \( D\vec{h} \).
4. Update \( \vec{h}(v) \) by \( \delta h(v) = -D\vec{h}(v)\delta t \).
5. Compute the Mass center \( \vec{c} = \int_{S^2} \vec{h} \sigma_M \); for all \( v \in M \), \( \vec{h}(v) = \vec{h}(v) - \vec{c} \); For all \( v \in M \), \( \vec{h}(v) = \frac{\vec{h}(v)}{|\vec{h}(v)|} \). This step makes the homeomorphism unique and keep the vertices on the mesh.
6. Compute the Harmonic energy \( E \)
7. If \( E - E_1 < \delta E \), return \( \vec{h}(v) \); otherwise, assign \( E \) to \( E_1 \) and repeat step 3 to step 6.
Through this algorithm we can get the unique homeomorphism $\overline{h}(\nu)$.

5. Experimental Results

We apply this algorithm to human brain surface mapping using C++. The 3D mesh is first extracted from 3D T1 weighted MRI image with matrix size 128x128x64 by using marching cubes algorithm after skull stripping (by using region growing algorithm [5]). The result shows below as figure 3.

![Figure 3. 3D MRI Triangulation](image)

The conformal mapping results are shown in Figure 4. From this example, we can see that the major features are mapped to the same position on the sphere. This suggests that the computed conformal mappings continuously depend on the geometry, and can match the major features consistently and reproducibly. In other words, conformal mapping is a good candidate for a canonical parameterization in brain mapping.

![Figure 4. Conformal Mapping](image)

References


[5]. Jong Geun Park, Chulhee Lee, Skull stripping based on region growing for magnetic resonance brain images, NeuroImage, Volume 47, Issue 4, 1 October 2009, Pages 1394-1407