Knots and Physics

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ABSTRACT

In this report, we introduce the basics of knots, knot polynomial invariants, and the Witten's functional integral, which show relationships with topics in theoretical physics, such as the statistical mechanics, quantum physics, etc.

1. Introduction

Knot theory is a theory studying the macroscopic physical phenomena of strings embedded in three-dimensional space. A fundamental problem in knot theory is determining when two knots are the same, which leads to the study of knot invariants, such as knot polynomial invariants. The knot invariants show a deep connections with the physical theories which demonstrate the physical principles working in the microscopic world, such as statistical physics and quantum physics. The content of this report is based on Kauffmann (L. H.,L).

In this brief introductory report, section 2 introduces the definitions of knots and knot equivalent relations (Reidmeister moves). Section 3 introduces the knot polynomial invariants, shows the relationship between the bracket polynomial and the statistical mechanics. Finally section 4 introduces Vassiliev invariants, Witten's functional integral and how they are related through Lie algebras.

2. Knots, Reidemeister moves

Definition 1. A **knot** is an embedding of a circle in 3-dimensional Euclidean space R^3. A knot is described by a planar diagram, i.e. a knot diagram. Figure 1 illustrates an example of a knot diagram of the trefoil knot.

A knot can be drawn in many different ways in knot diagrams. Different descriptions are regarded as equivalent if one knot can be transformed into the other through a continuous deformation (an ambient isotopy) in R^3 without cutting the string or passing through the string.

Definition 2. Ambient isotopy is the equivalence relation generated by the **Reidemeister moves**. Figure 2 demonstrates the three types (I, II, III) Reidemeister moves, which change the graphical structure of the knots without changing the topological types of the knots. The Reidemeister moves are useful in working out the behaviour of knot invariants.

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Fig. 1.— A knot diagram of the trefoil knot.

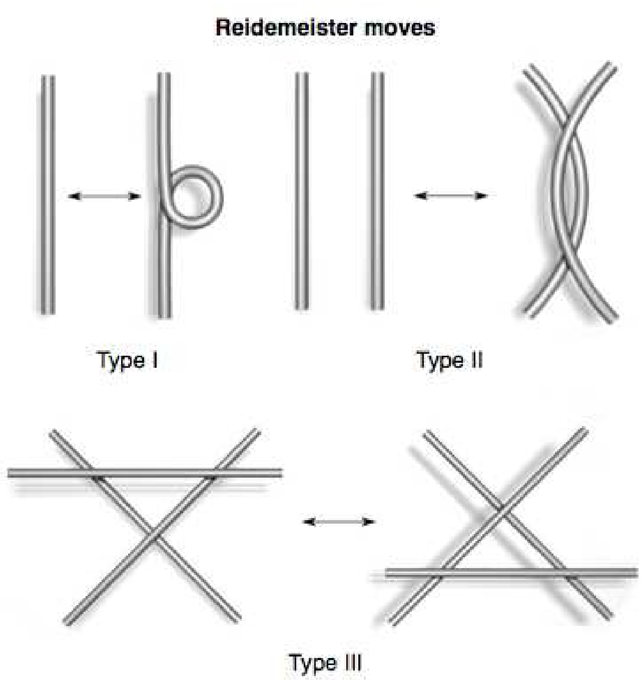

Fig. 2.— Three types Reidemeister moves.

3. Bracket Polynomials and Statistical Mechanics

A knot invariant is a "quantity" that is the same for equivalent knots. A knot polynomial is a knot invariant that is a polynomial. Well-known examples include bracket polynomials, Jones and Alexander polynomials, etc.

To define knot invariants, we first have a look the operations acting on the knot crossings. For a crossing, there are two types of operations splitting the crossing into two associated diagrams, as shown in Figure 3, type A split and type B split. A-region is defined as the region left to an observer walking along the undercrossing segments toward the crossing. In this way, any N-crossing knot K can be decomposed to 2^N final descendants without crossing, which are called states of K. Figure 4 shows the steps of splitting every crossing of the trefoil knot and the final descendants.

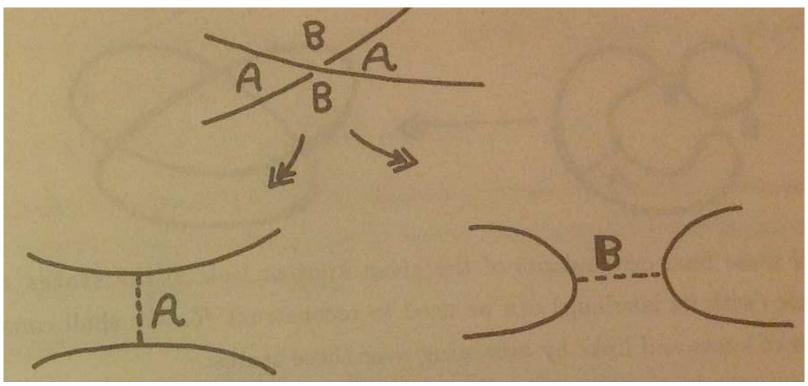

Fig. 3.— Type A and type B splitting.

Definition 3. **Bracket polynomial** is defined by the formula

$$\langle K \rangle = \sum_{\sigma} \langle K|\sigma\rangle d^{\|\sigma\|} \tag{1}$$

where σ denotes the final states, $\langle K|\sigma\rangle$ is the product of the labels (called vertex weights) attached to σ, and $\|\sigma\|$ represents the number of loops minus one in the final state. The bracket polynomial for the trefoil diagram is given as

$$\langle K \rangle = A^3 d^1 + 3A^2 B d^0 + 3AB^2 d^1 + B^3 d^2 \tag{2}$$

The bracket polynomial shows a relationship of the knot theory with physics. The bracket polynomial sums over all states, which is analogous to the partition function in statistical mechanics, summing over all states of the physical system with probability weighted individual states. There are other ways in connecting partition functions to knot invariants, which involves Hopf algebra, quantum groups, and local weights as the solutions to the Yang-Baxter equation. More information about this topic can be referred to Kauffmann (L. H.,L).

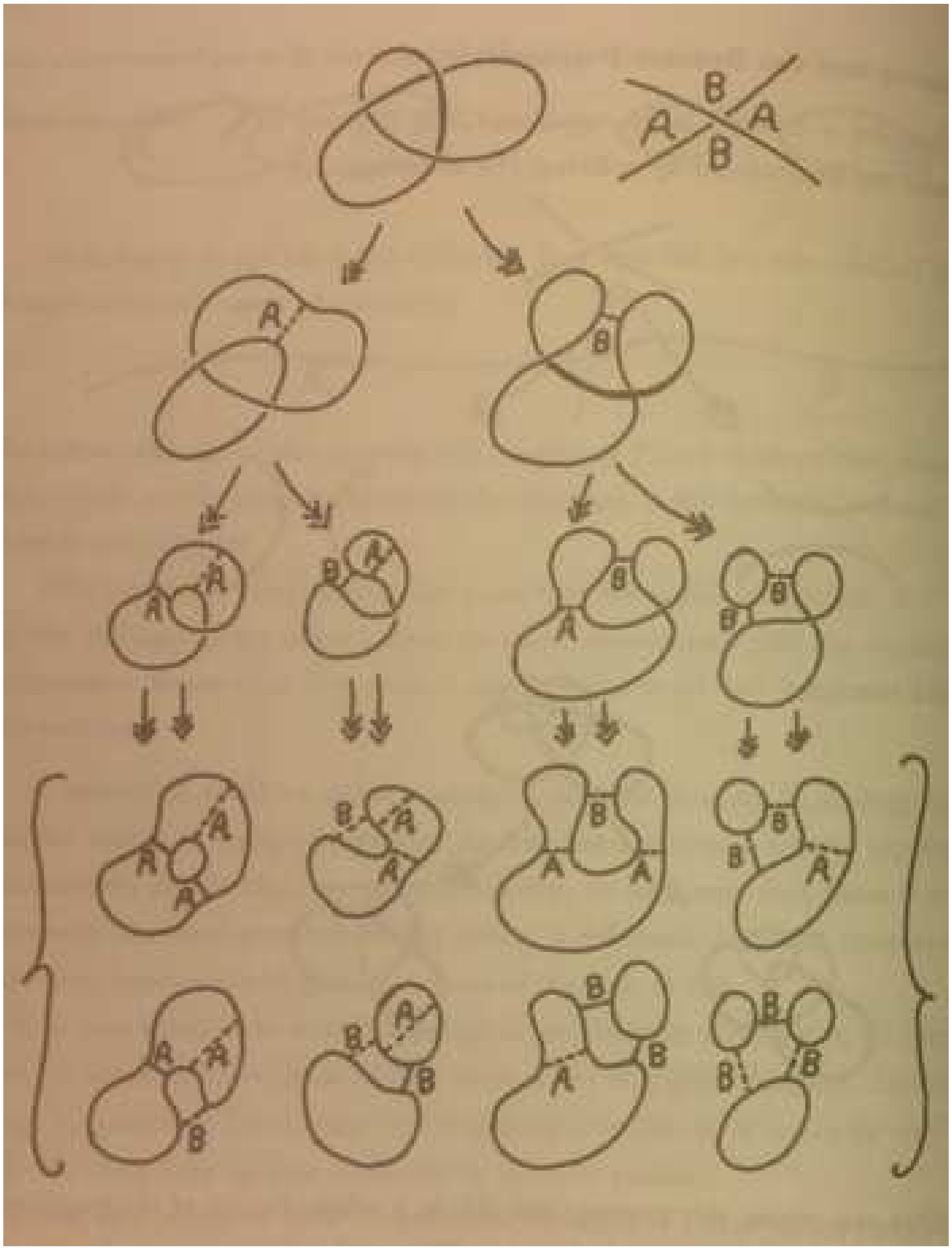

Fig. 4.— The decomposition of trefoil.

The bracket polynomial is invariant under regular isotopy (moves II and III) when $B = A^{-1}$ and $d = -A^2 - A^{-2}$ (Proof see Kauffmann (L. H.)). The invariant polynomial of ambient isotopy (moves I, II and III) can be obtained by a normalization using the writhe of K.

Definition 4. The **writhe** of K is defined as $w(K) = \sum_p \epsilon(p)$, which is the sum of the sign $\epsilon(p)$ of all of the crossings p in K. The definition of the sign is shown in Figure 5.

Definition 5. The bracket polynomial can be normalized to produce the polynomial invariant under all of the Reidemeister moves by the formula

$$L_K = (-A^3)^{-w(K)} \langle K \rangle \tag{3}$$

which is originally known as **Jones polynomial**.

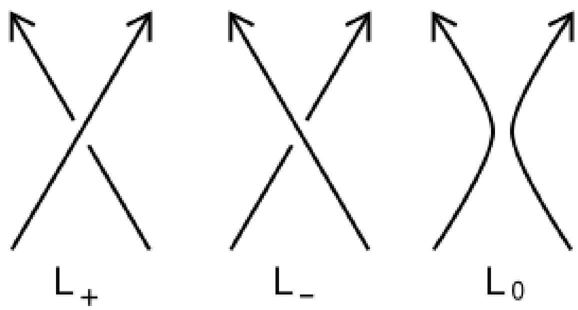

Fig. 5.— The sign convention of the crossing.

4. Vassiliev Invariants and Witten's Functional Integral

In this section we show the example of Lie algebras used to construct knot invariants, Vassiliev invariants, and its connection with Witten's functional integral, which ... in quantum field theory.

Definition 6: **Vassiliev invariant** is defined by the formula

$$V(K_*) = V(K_+) - V(K_-) \tag{4}$$

where K_+ denotes a positive crossing knot diagram, K_- denotes a negative crossing knot diagram, K_* denotes the knot diagram with the crossing replaced by a graphical node, and $V(K)$ is any knot invariant.

Figure 6 and Figure 7 demonstrates the four-term relation derived based on Vassiliev invariants from topology and from Lie algebra, i.e., the commutating relation

$$T^a T^b - T^b T^a = f_c^{ab} T^c \tag{5}$$

where $T^{a,b,c}$ are generators of Lie algebra and f_c^{ab} is the structure coefficients of Lie algebra. And hence illustrates the relationship between Vassiliev invariants and Lie algebras.

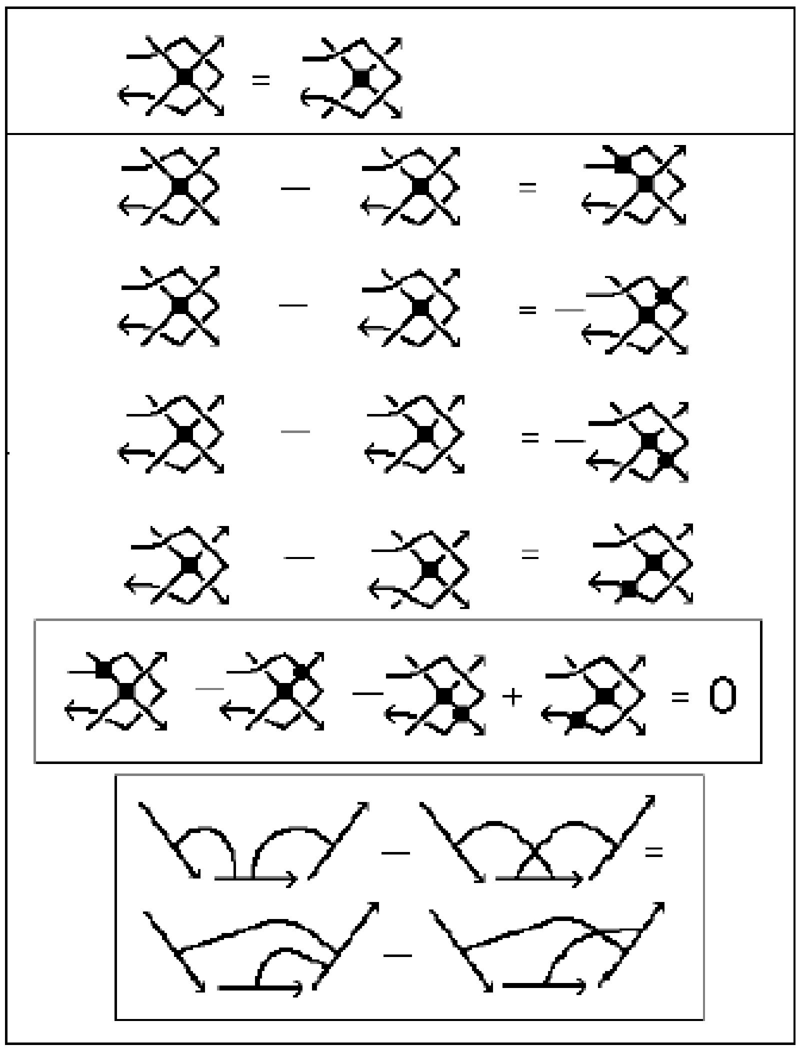

Fig. 6.— The four term relation from topology.

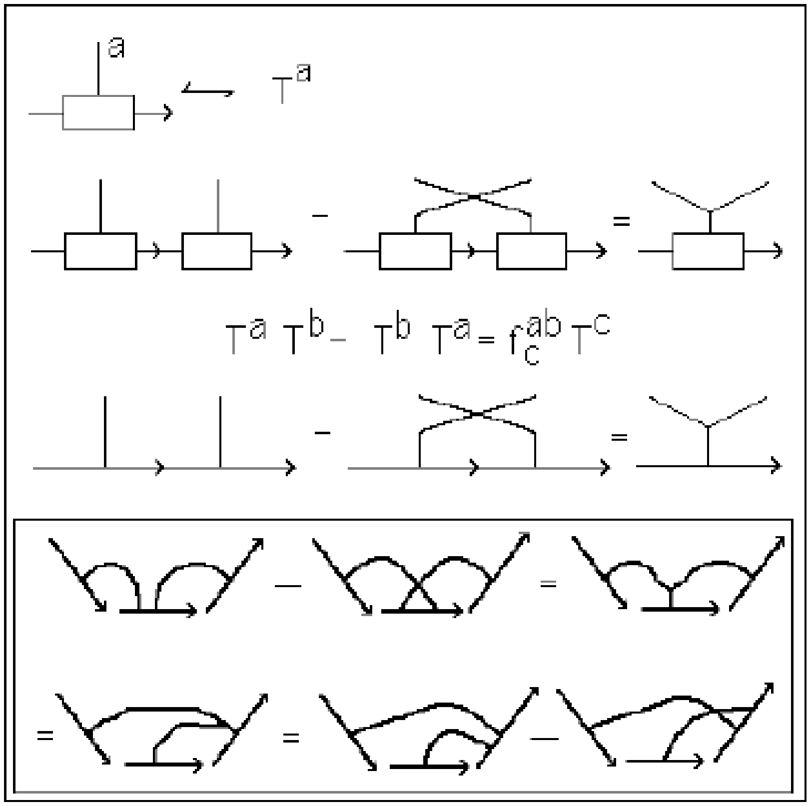

Fig. 7.— The four term relation from Lie algebra.

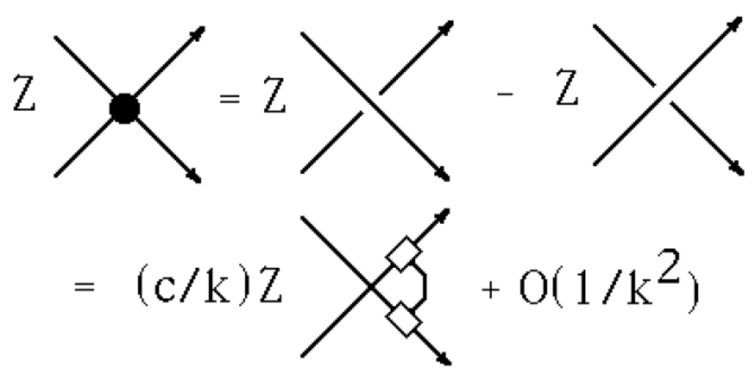

Fig. 8.—

Definition 7. **Witten's Functional Integral**

$$Z(M,K) = \int DAe^{(ik/4\pi)S(M,A)}tr(Pe^{\oint_K A}) = \int DAe^{(ik/4\pi)S}W_K(A) \tag{6}$$

where A is a gauge field defined on 3-manifold M, $S(M,A)$ is the Chern-Simons Lagrangian, and $W_K(A)$ is the Wilson loop integrating the gauge field A along a loop K (knot) in R^3 with transversal self-intersections.

How the Witten's functional integral denotes the topology invariants is demonstrated through the **theorem** examining the change of $Z(M,K)$ under an infinitesimal change of the loop K,

$$\delta Z(K) = (i4\pi/k)\int dAe^{(ik/4\pi)S(M,A)}\epsilon_{rst}dx_r dx_s dx_t T_a T_a W_K(A). \tag{7}$$

The result shows that $Z(K)$ is invariant if the variation of the loop does not create a local volume, i.e., $Z(K)$ is topologically invariant under regular isotopy (moves II and III).

In the case of switching a crossing (move I), the switching formula is

$$Z(K_+) - Z(K_-) = (i4\pi/k)\int DAe^{(ik/4\pi)S}T_a T_a\langle K_{**}|A\rangle = (i4\pi/k)Z(T^aT^aK_{**}) \tag{8}$$

where K_{**} denotes the result of replacing the crossing by a self-touching crossing, shown in Figure 8. The insertion of Lie algebra is exactly the weight assigned in the Vasilliev invariants. Hence, we can see the deep connection between the Vasilliev invariants and the Witten's functional integral.

REFERENCES

Kauffmann, L. H., 'Knot theory and Physics', http://www.ams.org/meetings/lectures/kauffman-lect.pdf

Kauffmann, L. H., 'Knots and Physics', World Scientific Publishers (1991)

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