The space of convex projective structures on surfaces

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Abstract

In this note we give a brief introduction to convex projective structures on manifolds, and discuss a convex-projective analogue of the Fricke-Klein theorem (due to Goldman).

1 Introduction

It is by this point a classical fact that finite-area surfaces with negative Euler characteristic may admit infinite families of distinct hyperbolic structures, that is, there are continuous families of non-conjugate deformations of the holonomy representations into PO(2, 1). The story is quite different for higher-dimensional hyperbolic manifolds, where the celebrated rigidity results of Mostow [Mos73] and Prasad [Pra73] indicate that such deformations into PO(n, 1) do not exist. It is then natural to ask whether these manifolds can potentially deform in a larger Lie group, like PGL(n + 1), and in this case the manifolds are said to be equipped with projective structures.

In this note, we give a brief introduction to projective structures on manifolds (with a particular focus on closed surfaces of negative Euler characteristic) and then an overview of the main ingredients for the proof of the following main theorem, due to Goldman [Gol90]:

Theorem ([Gol90]). Given a surface Σ with Euler characteristic χ(Σ) < 0, the deformation space of convex projective structures is homeomorphic to $\mathbb{R}^{-8\chi(\Sigma)}$.

1.1 Convex projective structures

Definition 1.1. An affine patch in $\mathbb{R}P^n$ is the set $\mathbb{R}P^n - \mathbb{P}(H)$ where $H$ is any hyperplane in $\mathbb{R}^{n+1}$.

Definition 1.2. A domain $\Omega \subset \mathbb{R}P^n$ is convex if the intersection of every line in $\mathbb{R}P^n$ with $\Omega$ is connected.

Definition 1.3. A convex domain $\Omega$ is properly convex if its closure is contained within an affine patch.

Definition 1.4. A point $p \in \partial \Omega$ is a strictly convex point if $p$ is not contained in a line segment of positive length in $\partial \Omega$. $\Omega$ is a strictly convex domain if it is properly convex and strictly convex at all points in $\partial \Omega$. 
There is a crucial distinction in the geometric behavior of properly convex domains and strictly convex domains: properly convex domains behave like symmetric spaces, whereas strictly convex domains behave more like manifolds with negative sectional curvature (see [CLT15] or [Bal14] for discussions on the matter).

The set of projective transformations of $\mathbb{RP}^n$ is $\text{PGL}(n+1,\mathbb{R}) := \text{GL}(n+1,\mathbb{R})/\mathbb{R} \times$. When $n$ is even, $\text{PGL}(n+1,\mathbb{R})$ is isomorphic to $\text{SL}(n+1,\mathbb{R})$, and when $n$ is odd, $\text{PGL}(n+1,\mathbb{R})$ is isomorphic to $\text{SL}^\pm(n+1,\mathbb{R})$ (the group of matrices with determinant $\pm 1$). Following along with the notation from [CLT15], for a convex domain $\Omega$, we denote by $\text{PGL}(\Omega)$ the subset of $\text{PGL}(n+1,\mathbb{R})$ preserving $\Omega$, and similarly for $\text{SL}^\pm(\Omega)$ in $\text{SL}^\pm(n+1,\mathbb{R})$.

**Definition 1.5.** A $(G, X)$-manifold has a (properly, resp. strictly) convex projective structure if $X = \Omega$ is some (properly, resp. strictly) convex domain in $\mathbb{RP}^n$ and $G = \text{PGL}(\Omega)$.

As is standard, a (properly, resp. strictly) convex projective manifold will refer to a manifold $M$ which arises as the quotient $\Omega/\Gamma$ where $\Omega$ is a (properly, resp. strictly) convex domain and $\Gamma < \text{PGL}(\Omega)$ acts properly-discontinuously and freely on $\Omega$. If $\Gamma$ has torsion, the quotient is called a (properly, resp. strictly) convex projective orbifold.

**Definition 1.6.** Two convex projective manifolds $M_1 = \Omega_1/\Gamma_1$ and $M_2 = \Omega_2/\Gamma_2$ are projectively equivalent if there exists a transformation $h \in \text{PGL}(n+1,\mathbb{R})$ for which $h(\Omega_1) = \Omega_2$ and $h\Gamma_1h^{-1} = \Gamma_2$.

**Proposition 1.7** (Proposition 1.3 in [CLT15]). Suppose $\Omega$ is properly convex and $\Gamma \leq \text{PGL}(\Omega)$. Then $\Gamma$ is a discrete subgroup of $\text{PGL}(n+1,\mathbb{R})$ if and only if $\Gamma$ acts properly discontinuously on $\Omega$.

**Definition 1.8.** Let $\Omega$ be properly convex and $x, y \in \Omega$. Let $p, q \in \partial\Omega$ be points on the line $(xy)$ ordered as in the picture below.

For a Euclidean metric $|\cdot|$ on $\mathbb{R}^{n+1}$, the Hilbert metric is given by

$$d_\Omega(x, y) = \frac{1}{2} \log \left( \frac{|p - y||q - x|}{|p - x||q - y|} \right)$$

This metric is invariant under the choice of lift of $p, x, y, q$, and so is well-defined on $\Omega$. When $\Omega$ is the interior of an ellipsoid, this metric exactly agrees with the usual Hyperbolic metric.

With this metric in mind, for a properly convex domain $\Omega$, all elements in $\text{PGL}(\Omega)$ act by isometries on $\Omega$ (but unless $\Omega$ is also strictly convex, it is not necessarily true that all isometries are contained in $\text{PGL}(\Omega)$. For the isometries in $\text{PGL}(\Omega)$, however, we have a familiar trichotomy:
Definition 1.9. Let $A \in \text{PGL}(\Omega)$ be an isometry.

- $A$ is elliptic if it is conjugate in $\text{PGL}(\Omega)$ into $\text{O}(n + 1)$.
- $A$ is parabolic if it is not elliptic and all of its eigenvalues have modulus 1.
- $A$ is hyperbolic otherwise.

2 Convex projective surfaces

Going forward, we will restrict attention to oriented closed surfaces of genus $g > 1$ so that we may think of deformation space $\mathcal{P}(S_g)$ as a convex projective analogue of the classical Teichmüller space. We aim to provide coordinates in the spirit of Fenchel–Nielsen [?] on the deformation space of convex-projective structures and use it to prove a convex-projective analog of the Fricke-Klein theorem [FK65]. The assumption that the surface is closed is merely a convenience for exposition; indeed one can extend this notion to compact surfaces with boundary as well (see [Go90]).

The strategy is as follows: we find a pants decomposition of a surface $S_g$ and analogs of the “length” and “twist” parameters from the classical Fenchel-Nielsen coordinates; we also find the space of deformations on the interior of each pair of pants. All of this culminates into the main theorem

Theorem ([Go90]). The deformation space of a convex projective structures on $S_g$ is $(16g - 16)$-dimensional.

2.1 Deformations

Definition 2.1. Given a closed convex projective surface $S_g$, the deformation space of convex projective structures on $S_g$ is

$$\mathcal{P}(S_g) := \text{Hom}_{\text{DF}}(\pi_1(S_g), \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R}),$$

where $\text{Hom}_{\text{DF}}$ denotes the discrete, faithful representations and the quotient action is conjugation on the image of the representations.

Definition 2.2. A hyperbolic transformation $A \in \text{SL}(3, \mathbb{R})$ is positive if it has 3 distinct, real, positive eigenvalues $\lambda, \mu, \nu$.

We denote by $\text{Hyp}_+$ the set of positive hyperbolic transformations. We order the the eigenvalues $0 < \lambda < \mu < \nu$. With this ordering, the eigenvector corresponding to $\lambda$ is the repelling fixed point and the eigenvector corresponding to $\nu$ is the attracting fixed point. The principal line of a positive hyperbolic transformation is the unique line in $\mathbb{RP}^2$ passing through the repelling and attracting fixed points.

Lemma 2.3. For $A \in \text{Hyp}_+$,

$$\ell(A) = \log\left(\frac{\nu}{\lambda}\right) \quad \text{and} \quad m(A) = 3 \log(\mu)$$

are $\text{SL}(3, \mathbb{R})$-conjugacy class invariants of $A$. 

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Note that $\ell(A) > 0$ and $|m(A)| < \ell(A)$, and we’ll let $\mathcal{U} \subset \mathbb{R}^2$ be the set of admissible values $(\ell, m)$.

**Theorem 2.4** (See Theorem 3.2 in [Go90]). Let $S_g = \Omega/\Gamma$ be a closed convex projective surface with genus $g \geq 1$.

1. $\Omega$ is a strictly convex domain.

2. For any nontrivial $\gamma \in \Gamma$, then $\gamma \in \text{Hyp}_+$. Moreover, every homotopically nontrivial closed curve in $S_g$ is freely homotopic to a unique (principal) closed geodesic.

In fact, for any positive hyperbolic element $\gamma \in \pi_1(S_g)$, $\ell(\gamma)$ is the Hilbert length of the closed geodesic in the same free homotopy class. In this way, the pair $(\ell(A), m(A))$ provide “length parameters” analogous to those of the Fenchel–Nielsen coordinates.

### 2.2 Convex projective structures on pairs of pants

Let $P$ be a pair of pants with boundary geodesics $A$, $B$, and $C$. Given three arcs $a, b, c$ such that $a$ spirals outward from $B$ to $C$, $b$ spirals outward from $C$ to $A$, and $c$ spirals outward from $A$ to $B$ (see Figure 1), we can decompose $P$ into two ideal triangles, $T_0$ and $T_1$.

![Figure 1: Pair of pants and a collection of arcs](image)

Examining this decomposition in the universal cover, we see that choosing a particular lift of $T_0$ and 3 suitably-chosen lifts of $T_1$, we obtain a hexagon in where the deck transformation $A$ sends $\tilde{T}_{1,b}$ to $\tilde{T}_{1,c}$, $B$ sends $\tilde{T}_{1,c}$ to $\tilde{T}_{1,a}$, and $C$ sends $\tilde{T}_{1,a}$ to $\tilde{T}_{1,b}$ (see Figure 2). After an appropriate choice of developing and holonomy maps, $\text{dev}(\tilde{T}_0 \cup \tilde{T}_{1,a} \cup \tilde{T}_{1,b} \cup \tilde{T}_{1,c})$ is a convex hexagon in $\Omega \subset \mathbb{R}^2$ and $\text{hol}(A), \text{hol}(B), \text{hol}(C) \in \text{Hyp}_+$ (abusing notation slightly, we’ll identify these triangles and the deck transformations with their images under the developing and holonomy maps).
In this way, one can assign a convex projective structure to any septuple of four convex triangles in $\Omega$ and three positive hyperbolic transformations $(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C)$ satisfying

1. $\overline{\triangle_a}, \overline{\triangle_b}, \overline{\triangle_c}$ intersect $\overline{\triangle_0}$ along its three sides
2. $\overline{\triangle_0} \cup \overline{\triangle_a} \cup \overline{\triangle_b} \cup \overline{\triangle_c}$ is a convex hexagon
3. $ABC = 1$
4. $A(\triangle_b) = \triangle_c$, $B(\triangle_c) = \triangle_a$, $C(\triangle_a) = \triangle_b$
5. The vertices of $\overline{\triangle_0}$ (and thus the intersections of the closures of the adjacent triangles) are the repelling fixed points of $A, B, C$.

Denote the set of all such tuples by $\mathfrak{T}$.

Choosing homogeneous coordinates for the vertices of the hexagon, as in Figure 2 and performing a serious of tedious (and unenlightening) calculations, one obtains the following:

**Theorem 2.5** (Proposition 4.3 in [Gol90]). $\mathfrak{T}$ is 8-dimensional with 6 of these dimensions corresponding to the “length parameters”. That is,

$$\mathfrak{T} \to \mathfrak{U}^3$$

$$(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C) \mapsto (\ell(A), m(A), \ell(B), m(B), \ell(C), m(C))$$

is a fibration with fibers homeomorphic to $\mathbb{R}^2$. 
2.3 Twist parameters

Let $C$ be simple closed geodesic in $S_g$ with associated hyperbolic element $A \in \text{Hyp}_+^+$ (we assume that $A = \text{diag}(\lambda, \mu, \nu)$). It is easy to see that the two-parameter subgroup of $\text{SL}(3, \mathbb{R})$ generated by

$$U(s) = \begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^s \end{pmatrix} \quad \text{and} \quad V(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

commutes with $A \in \text{Hyp}_+^+$; moreover, $U(s)$ and $V(t)$ generate the entire centralizer of $A$. As such, for a fixed deformation $\rho$, post-composition by an element of $\langle U(s), V(t) \rangle$ gives rise to a new deformation (and thus a new convex projective structure) on $S_g$ under which $C$ is invariant. This yields a 2-parameter analogue of the classical “twist parameters” of Fenchel and Nielsen.

2.4 Putting it all together

We consider a disjoint collection of $3g - 3$ simple closed geodesics on $S_g$ that provide a decomposition into $2g - 2$ pairs of pants. From the previous section, we see that to each simple closed curve we can associate a 2-dimensional family of deformations coming from the twist parameters and another 2-dimensional family of deformations coming from the length parameters. From Theorem 2.5 to the interior of each pair of pants we can associate another 2-dimensional family of deformations.

**Theorem 2.6 ([Gol90]).** The deformation space of a convex projective structures on $S_g$ is $(16g - 16)$-dimensional.

**Proof.**\[2(3g - 3) + 2(3g - 3) + 2(2g - 2) = 16g - 16.\]
3 References


