Hyperbolic Coxeter groups of rank 4

Younghwan Kim

Spring 2016

1. Introduction

The standard classification of finite reflection groups and affine Coxeter systems can be found in [1]. Humphrey presented a characterization to classify hyperbolic Coxeter systems. In this paper, we will state and use the characterization to determine the hyperbolic Coxeter groups of rank 4, which is an exercise in [1].

2. Preliminaries

Let \((W, S)\) be an irreducible Coxeter system where \(m(s_i, s_j)\) represents the order of \(s_i s_j\) for all \(i, j\). Let \(B\) be a symmetric bilinear form. Let \(A\) be the matrix of the bilinear form, in other words, \(A = (a_{ij})\) is a symmetric matrix with \(a_{ij} = -\cos(\pi / m(s_i, s_j))\). We let \(\Gamma\) be the associated Coxeter graph, which is undirected, labeled graph with vertex set \(S\) and edges \(s_i s_j\) when \(m(s_i, s_j) \geq 3\) and each edge labelled by the order \(m(s_i, s_j)\). If \(\Gamma\) is a connected graph, then we say irreducible.

For any real symmetric matrix over a finite dimensional vector space, it is called positive definite (resp. positive semidefinite) if all eigenvalues of the matrix are positive (resp. nonnegative). We also say that the matrix is of positive type if it is positive semidefinite, including positive definite. We also call \(\Gamma\) positive definite or positive semidefinite when the associated matrix \(A\) (or bilinear form \(B\)) has the corresponding property. We define the irreducible Coxeter system \((W, S)\) to be hyperbolic if \(B\) has signature \((n - 1, 1)\) for some \(n > 1\) and \(B(v, v) < 0\) for all \(v \in C\) where \(C\) is the cone \(C = \{v \in V : B(v, \alpha_s) > 0\}\) for all \(s \in S\) and \(\alpha_s\) is the simple root corresponding to \(s \in S\). We can find the following characterization of hyperbolic Coxeter systems in [1].

Proposition 1 [1, Proposition 6.8]. Let \((W, S)\) be an irreducible Coxeter system, with graph \(\Gamma\) and associated bilinear form \(B\). It is hyperbolic if and only if the following conditions are satisfied:

(a) \(B\) is nondegenerate, but not positive type \(^1\).
(b) For each \(s \in S\), the Coxeter graph obtained by removing \(s\) from \(\Gamma\) is of positive type.

3. Hyperbolic Coxeter groups of rank 4

There are 6 undirected simple graphs on 4 vertices: (1) path, (2) star, (3) square, (4) star and one edge, (5) one edge removed from the complete graph \(K_4\), and (6) the complete graph \(K_4\).

\(^1\)In [1] it is positive definite, but I highly suspect that it is an error because in the proof of the proposition the author assumes that it is not positive type.
3.1 Path

Suppose \( \Gamma \) is a path, and label its three edges by \( l,m,n \geq 3 \). Let \( a = \cos(\frac{\pi}{l}) \), \( b = \cos(\frac{\pi}{m}) \), \( c = \cos(\frac{\pi}{n}) \). The matrix of the form \( B \) is then

\[
\begin{bmatrix}
1 & -a & 0 & 0 \\
-a & 1 & -b & 0 \\
0 & -b & 1 & -c \\
0 & 0 & -c & 1
\end{bmatrix}
\]

Its characteristic polynomial is \( (1-x)^4 - (a^2 + b^2 + c^2)(1-x)^2 + a^2c^2 \). Thus, the eigenvalues are

\[
1 \pm \sqrt{a^2 + b^2 + c^2 \pm \sqrt{(a^2 + b^2 + c^2)^2 - 4a^2c^2}}.
\]

To satisfy the first condition of Proposition 1, there must exist a negative eigenvalue. Notice that all eigenvalues are nonnegative when \( (l,m,n) = (3,3,3), (3,3,4), (3,3,5), (3,4,3), (4,3,4) \). Thus, we need to consider all possibilities \( (l,m,n) \) for \( 3 \leq l, m, n \leq 6 \) except \( (l,m,n) = (3,3,3), (3,3,4), (3,3,5), (3,4,3), (4,3,4) \). For the second condition of Proposition 1, the Coxeter graph obtained by removing a vertex from \( \Gamma \) must be positive type, which means that all eigenvalues of the corresponding matrix should be nonnegative. Observe first that if we take away the second or the third vertex, then the remained graph is union of two irreducible positive type Coxeter graph, which is positive type. So, consider a path on three vertices which is obtained by removing a degree one vertex, and label its two edges by \( l,m \geq 3 \):

The matrix of the form is then

\[
\begin{bmatrix}
1 & -a & 0 \\
-a & 1 & -b \\
0 & -b & 1
\end{bmatrix}
\]

Its characteristic polynomial is \( (x-1)(x^2 - 2x + 1 - a^2 - b^2) \), thus the eigenvalues are \( 1, 1 \pm \sqrt{a^2 + b^2} \). These are all nonnegative when \( a^2 + b^2 \leq 1 \), since the cosines ranges in value from 1/2 to 1, there are eight possibilities: \( (l,m) = (3,3), (3,4), (3,5), (3,6), (4,3), (4,4), (5,3), (6,3) \). Therefore, we have 10 possibilities: \( (l,m,n) = (4,3,5), (3,5,3), (5,3,5), (4,4,3), (4,4,4), (4,3,6), (4,3,6), \ldots \)
(5,3,6), (3,3,6), (3,6,3), (6,3,6), and we have following 10 hyperbolic Coxeter graphs of paths up to isomorphism.

3.2 Star

Suppose $\Gamma$ is a star, and label its three edges by $l, m, n \geq 3$. Let $a = \cos(\pi/l), b = \cos(\pi/m), c = \cos(\pi/n)$. The matrix of the form $B$ is then

\[
\begin{pmatrix}
1 & -a & -b & -c \\
-a & 1 & 0 & 0 \\
-b & 0 & 1 & 0 \\
-c & 0 & 0 & 1 \\
\end{pmatrix}
\]

Its characteristic polynomial is $(1 - x)^4 - (a^2 + b^2 + c^2)(1 - x)^2$. Thus, the eigenvalues are $1, 1, 1 \pm \sqrt{a^2 + b^2 + c^2}$. To satisfy the first condition of Proposition 1, we have $a^2 + b^2 + c^2 > 1$. For the second condition, if we remove the vertex of degree three, then the remained graph is positive definite. If we remove any leaf (any vertex of degree 1), then we get a path on three vertices, and we have already cared of this case. Therefore we have 4 possibilities: $(l, m, n) = (3,3,5), (3,3,6), (3,4,4), (4,4,4)$, and we have following 4 hyperbolic Coxeter graphs of stars up to isomorphism.

3.3 Square
Suppose $\Gamma$ is a square, and label its four edges by $l, m, n, p \geq 3$. Let $a = \cos(\pi/l), b = \cos(\pi/m), c = \cos(\pi/n), d = \cos(\pi/p)$. The matrix of the form $B$ is then

$$
\begin{bmatrix}
1 & -a & 0 & -d \\
-a & 1 & -b & 0 \\
0 & -b & 1 & -c \\
-d & 0 & -c & 1 \\
\end{bmatrix}
$$

The determinant is nonpositive, since $a, b, c, d \geq 1/2$, and is 0 just when $a = b = c = d = 1/2$ or $l = m = n = p = 3$. Thus, the first condition of Proposition 1 holds for all 4-tuples except $(3,3,3,3)$. For the second condition, if we remove any vertex of the square, then we get a path on three vertices, and we have already cared of this case. Therefore we have 12 possibilities: $(l, m, n, p) = (3,3,3,4), (3,3,3,5), (3,3,3,6), (3,3,4,4), (3,4,3,4), (3,4,3,5), (3,4,3,6), (3,4,4,4), (3,5,3,5), (3,5,3,6), (3,6,3,6), (4,4,4,4)$, and we have following 12 hyperbolic Coxeter graphs of squares up to isomorphism.

![Coxeter graphs of squares up to isomorphism](image-url)

### 3.4 Star and one edge

Suppose $\Gamma$ is a star and one edge, and label its four edges by $l, m, n, p \geq 3$. Let $a = \cos(\pi/l), b = \cos(\pi/m), c = \cos(\pi/n), d = \cos(\pi/p)$. The matrix of the form $B$ is then

$$
\begin{bmatrix}
1 & -a & -c & -d \\
-a & 1 & -b & 0 \\
-c & -b & 1 & 0 \\
-d & 0 & 0 & 1 \\
\end{bmatrix}
$$

The determinant is negative since $a, b, c, d \geq 1/2$, and the first condition holds for all $3 \leq l, m, n, p \leq 6$. For the second condition, if we remove any vertex of $\Gamma$, then we get either: (1) a path on three vertices, (2) union of $K_2$ and a vertex, or (3) a triangle. Consider a triangle, and label its three edges by $l, m, n \geq 3$:
The matrix of the form is then
\[
\begin{pmatrix}
1 & -a & -c \\
-a & 1 & -b \\
-c & -b & 1
\end{pmatrix}
\]

The determinant is nonpositive, since \(a, b, c \geq 1/2\), and is 0 just when \(a = b = c = 1/2\) or \(l = m = n = 3\). Therefore we have 4 possibilities: \((l, m, n, p) = (3, 3, 3, 3), (3, 3, 3, 4), (3, 3, 3, 5), (3, 3, 3, 6)\), and we have following 4 hyperbolic Coxeter graphs of star and one edge up to isomorphism.

\[
\begin{array}{cccc}
\text{4} & \text{5} & \text{6} \\
\end{array}
\]

3.5 Complete graph \(K_4\) and \(K_4 \setminus \{e\}\)

Suppose \(\Gamma\) is either the complete graph \(K_4\) or \(K_4 \setminus \{e\}\) for some edge \(e\) respectively, and label its six (resp. five) edges by \(l, m, n, p, r, s \geq 3\) (resp. \(l, m, n, p, r \geq 3\)). The matrix of the form \(B\) is then
\[
\begin{pmatrix}
1 & -a & -c & -d \\
-a & 1 & -b & -e \\
-c & -b & 1 & -f \\
-d & -e & -f & 1
\end{pmatrix}
\]
or resp.
\[
\begin{pmatrix}
1 & -a & -c & -d \\
-a & 1 & -b & -e \\
-c & -b & 1 & 0 \\
-d & -e & 0 & 1
\end{pmatrix}
\]
The determinant is negative since \(a, b, c, d, e, f \geq 1/2\) (resp. \(a, b, c, d, e \geq 1/2\)), hence the first condition holds. For the second condition, if we remove any vertex of \(\Gamma\), then we get a triangle (resp. either a path on three vertices or a triangle). Thus, we have following hyperbolic Coxeter graphs up to isomorphism.

Reference


