Hyperbolic Geometry on the Figure-Eight Knot Complement

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Hyperbolic Space

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We can use the upper half space model of hyperbolic space: $\mathbb{H}^n := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ with metric $\|x\|_H := \frac{\|x\|_E}{x_n}$, where $\|\cdot\|_E$ is the standard Euclidean metric.
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Figure: Picture Credit: Wikipedia.org
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with metric \( \|x\|_H := \frac{\|x\|_E}{x_n} \), where \( \|\cdot\|_E \) is the standard Euclidean metric.

The geodesics in this space are arcs of circles perpendicular to the plane \( x_n = 0 \) and vertical lines.

Figure: Picture Credit: Wikipedia.org
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Hyperbolic $n$-manifolds

Hyperbolic $n$-manifolds are the Riemannian manifolds such that each point has a neighborhood isometric to an open subset of $\mathbb{H}^n$. 
In this talk I will show that the figure-eight knot complement is a hyperbolic manifold.
Gluing Two Tetrahedra

- We can glue the following tetrahedra together to get a cell complex $M$. The vertices of the tetrahedra are all mapped to a single vertex $v$. We saw in class that $M$ is not a manifold because every neighborhood of $v$ has a neighborhood homeomorphic to a cone on a torus.
However, $M \setminus \{v\}$ is a manifold (it is easy to see that every other point on $M$ has a Euclidean neighborhood).
Gluing Two Tetrahedra

**Theorem**

\( M \setminus \{v\} \) is homeomorphic to \( S^3 - L \), where \( L \) is the figure-eight knot.
Construcfing a homeomorphic cell complex

- Define a 1-complex $K^1 \subseteq S^3$ using the following diagram:

Figure: The cell complex $K^1$. 
Constructing a homeomorphic cell complex

- Attach four 2-cells to $K^1$ to get a 2-complex $K^2$.

Figure: The 2-cells $A$, $B$, $C$, and $D$. 
There is a homeomorphism $F$ taking $S^3 \setminus K^1$ to the complement of the following cell complex $K^1_1$:

![Cell Complex $K^1_1$]

**Figure:** The cell complex $K^1_1$
The homeomorphism $F$

$F$ maps neighborhoods of the 1-complexes 1 and 2 as shown in this diagram and leaves the rest of $S^3 \setminus K^1$ unchanged.
This homeomorphism takes the four 2-cells of $K^2$ ($A, B, C, \text{ and } D$) to the following diagram. Since the image of $A \cup B \cup C \cup D$ under the homeomorphism $F$ is a plane, the complement to $S^3 \setminus K^2$ is two open 3-balls.
We can view these open 3-balls as the interior of two 3-cells in order to extend $K^2$ to a cell complex $K^3$ for $S^3$. The boundaries of the 3-cells are attached to $K^2$ according to the following diagram.
The 0-cells and the 1-cells 3, 4, 5, and 6 collectively form the figure-eight knot $L$. By identifying all of these to a single point $x$ and then removing $x$ we get a space homeomorphic to $S^3 \setminus L$, the figure-eight knot complement.
All that remains is to show that the cell complex that left over after collapsing these points is our original cell complex $M$ obtained by gluing two tetrahedron.
Hyperbolic Polyhedron

Definition

A hyperbolic ideal polyhedron is a subset of $\mathbb{H}^n$ that is the intersection of a finite collection of half spaces in $\mathbb{H}^n$ whose vertices are all on $S^{n-1}_{\infty}$. A facet of a polyhedron $P$ is the intersection of $P$ with a codimension one hyperplane $T$ such that exactly one component of $\mathbb{H}^n \setminus T$ is disjoint from $P$.

Figure: Diagram credited to Vladimir Bulatov, Oregon State University.
Putting a hyperbolic geometry on $S^3 \setminus L$

Lemma

Let $F_1, F_2,$ and $F_3$ be three facets of an ideal tetrahedron in $\mathbb{H}^3$. Let $\beta_{12}$ be the interior angle between $F_1$ and $F_2$ and define $\beta_{23}, \beta_{31}$ similarly. Then $\beta_{12} + \beta_{23} + \beta_{31} = \pi$.

Proof:

Since these three facets have a point at infinity, they are all Euclidean planes in the upper half space $\mathbb{H}^3$. Then $\beta_{12}, \beta_{23},$ and $\beta_{31}$ form the interior angles of a Euclidean triangle.
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- Note then that if our ideal tetrahedron is regular (symmetric) then each of the angles $\beta_{ij} = \pi/3$. 
Putting a hyperbolic geometry on $S^3 \setminus L$

**Lemma**

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- Note then that if our ideal tetrahedron is *regular* (symmetric) then each of the angles $\beta_{ij} = \pi/3$.
- For each point $x$ of $M$ lying in a one cell the two cells of $M$ are glued to $x$ six times. Therefore the six interior angles around $x$ sum to $2\pi$ and so every point of $M \setminus \{v\}$ has a hyperbolic neighborhood.
References


