Hyperbolic Geometry on the Figure-Eight Knot Complement

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1 Introduction and History

The exact relationship between knot theory and non-euclidean geometry was a puzzle that survived more than 100 years. The histories of the two subjects were clearly intertwined; Carl Friedrich Gauss was a pioneer and did much to popularize both fields.

After Gauss their two histories diverge a little, although taken together the list of luminaries in the two fields reads like a mathematical and scientific “Who’s Who?” of the 19th and early 20th century. Nikolai Lobachevsky, Felix Klein, and Henri Poincaré were three of the main developers of non-euclidean geometry while William Thomson (Lord Kelvin), James Clerk Maxwell, and Emil Artin were instrumental in fleshing out knot theory[4]. Although both subjects came to be recognized as following under the umbrella term “topology” and although giants such as Bernhard Riemann and Max Dehn [4, 7] worked in both areas the two fields remained disjoint until they were finally reunited in the 1970’s.

In 1973 Robert Riley, then a graduate student at the University of Southampton in England, succeeded in showing that the figure-eight knot complement had a hyperbolic structure [4]. He did this by first showing that since fundamental group of the figure-eight knot complement is isomorphic to a subgroup of PSL$_2$C, and then using the theory of Haken (of four color theorem fame) manifolds to show that the figure-eight knot complement is homeomorphic to $\mathbb{H}^3$ mod a discrete group of isometries [6]. Riley later showed that several other knot complements admit a hyperbolic structure and conjectured that indeed all knot complements except for torus and satellite knots admit a hyperbolic structure.

It seems to often be the case in mathematics that the best way to make progress on a subject is to interest someone else in it. In 1977 Riley did exactly this when he met William Thurston at Princeton and motivated him to start investigating hyperbolic structures on knot complements [4]. Thurston soon came up with a more explicit way of showing that the figure-eight knot complement is hyperbolic. It is Thurston’s construction, which starts by gluing two tetrahedra together, that we will follow in Section 2. Relying in part on his experiences with knot complements [9] in 1978 Thurston completed his “hyperbolization theorem” or “geometrization theorem” (note that this is is a special case of the “geometrization
conjecture” that Perelman famously proved much later) for which he won a Fields Medal in 1982 [4]. Although Thurston never published his proof (the reasons for which he explains in [9]) this theorem quickly entered the mainstream and confirmed Riley’s conjecture that almost all knot complements are hyperbolic.

# 2 The Figure-Eight Knot Complement

## 2.1 Hyperbolic Geometry

Hyperbolic geometry is a non-euclidean geometry (some use the terms non-euclidean geometry and hyperbolic geometry as synonyms) discovered independently in the early 19th century by Gauss, Bolyai, and Lobachevsky [1]. Historically non-euclidean geometry arose from trying to prove that the following statement equivalent to Euclid’s fifth postulate was implied by the first four postulates: “Given a line and a point not on it, there is exactly one line going through the given point that is parallel to the given line” [1]. Efforts at deriving a contradiction from assuming the negation of this statement ended up defining the basis of a different (hence non-euclidean) but logically consistent geometry. In the late 1830’s Lobachevsky “suggested that curved surfaces of constant negative curvature might represent non-euclidean geometry” a suggestion that Ferdinand Minding proved soon after [1]. Minding’s proof informs the basis of our modern understanding of hyperbolic space:

**Definition 1** Hyperbolic space $\mathbb{H}^n$ is the unique complete simply-connected Riemannian $n$-manifold with all sectional curvatures being $-1$.

There are several “models” used to study hyperbolic space. Although any one model would be sufficient to develop the theory of hyperbolic geometry, each model has inherent advantages and disadvantages and going back and forth between the various models is often the most efficient way to work with hyperbolic geometry. In this space we will use the “upper half-space” model:

**Definition 2** $\mathbb{H}^n := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ with metric $ds_H^2$ given by

$$ds_H^2 := \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{x_n^2}.$$ 

In this space the geodesics are either vertical lines (lines in the $x_n$ direction) or arcs of circles that intersect the plane $x_n = 0$ at right angles as shown in Figure 1. Finally, we can define hyperbolic manifolds to be the subset of Riemannian manifolds such that the metric in a neighborhood of each point “looks like” our model of hyperbolic space:

**Definition 3** Hyperbolic $n$-manifolds are the Riemannian manifolds such that each point has a neighborhood isometric to an open subset of $\mathbb{H}^n$.  

2
2.2 Knots and Knot Complements

Knots, once proposed by Helmholtz and studied by Lord Kelvin as a model of an atom have seen a renewed interest in recent years with the proliferation of string theory that postulates knots may be even more fundamental physical structures [7].

Definition 4 A mathematical knot is an embedding of the circle $S^1$ into three-dimensional euclidean space $\mathbb{R}^3$.

A knot complement is normally defined to be the complement of the knot in a surrounding 3-sphere. The figure-eight knot, which as Thurston notes [8] is the second most commonly occurring knot in garden hoses and vacuum cords, is shown in Figure 2. In the remainder of this paper we will prove the following theorem

Theorem 1 (Riley 1973, Thurston 1978) The figure-eight knot complement is a hyperbolic manifold.

following the proof in [5] (which in turn uses Thurston’s construction) relatively faithfully.

Consider the two tetrahedra represented in Figure 3 with corresponding faces identified to form a space $M$. That is, each face on the left tetrahedron has a unique pattern of single and double arrows that is identified with the corresponding face on the right tetrahedron in the obvious way.

It is a simple job of vertex chasing to show that all of the vertices are identified to a single point $v$. A neighborhood of $v$ is homeomorphic to a torus and so $M$ is not a manifold. However every point on $M$ besides the unique vertex has a euclidean neighborhood in $\mathbb{R}^3$ and so $M \setminus \{v\}$ is a 3-manifold.

Theorem 2 $M \setminus \{v\}$ is homeomorphic to $S^3 \setminus L$, where $L$ is the figure-eight knot.

To prove Theorem 2 we will start with the figure-eight knot $L$ and uses a chain of homeomorphic cell complexes to end up with $M \setminus \{v\}$.
2.3 Constructing an Equivalent CW-Complex

Define a 1-complex $K^1 \subseteq S^3$ using the diagram in Figure 4 and attach four 2-cells $A, B, C,$ and $D$ to $K^1$ to get a 2-complex $K^2$. 
Figure 4: The cell complex $K^1$.

![Figure 4: The cell complex $K^1$.](image)

Figure 5: The 2-cells $A, B, C,$ and $D$.

![Figure 5: The 2-cells $A, B, C,$ and $D$.](image)

Figure 6: The cell complex $K^1_1$

There is a homeomorphism $F$ taking $S^3 \setminus K^1$ to the complement of the cell complex $K^1_1$ shown in Figure 6.

![Figure 6: The cell complex $K^1_1$.](image)

Figure 7: The homeomorphism $F$

This homeomorphism maps neighborhoods of the 1-complexes 1 and 2 as shown in this diagram and leaves the rest of $S^3 \setminus K^1$ unchanged as shown in Figure 7. $F$ takes the 2-cells of $K^2$ to the two cells shown in Figure 8 to form a 2-complex $K^2_1$. Since the union of these cells is a plane, the complement of the complex in Figure 8 is two open 3-balls.

![Figure 7: The homeomorphism $F$.](image)
If we view these two open 3-balls as the interior of two 3-cells then these 3-cells are attached to the boundary of $K_1^2$ as shown in Figure 9 and defines a cell complex $K^3$.

All of the 0-cells and the 1-cells 3,4,5, and 6 (all except the 1-cells 1 and 2 which we added) combine to form the figure-eight knot $L$. Therefore, collapsing all of these cells to a single point $v$ and then removing that point is equivalent to taking the complement of $L$ as a subset of $S^3$. Therefore all that remains is to show that this cell complex is the same as the cell complex $M$ that we got by gluing two tetrahedra together. It has the correct number of cells of each dimension and the attaching maps for the 2 and 3-cells implied by Figures 8 and 9 “are readily seen to give the required cell complex for $M$” [5]. Indeed, comparing each face or edge to the identification shown in Figure 3 shows that the two cell complexes are the same.

2.4 Hyperbolic Geometry on the Figure-Eight Knot Complement

In the previous section we constructed $M \setminus \{v\}$ and showed it to be homeomorphic to $S^3 \setminus L$. We will now show that this space is hyperbolic. First we need a few definitions.

**Definition 5** An ideal polyhedron is a subset of $\mathbb{H}^n$ that is the intersection of a finite collection of half spaces in $\mathbb{H}^n$ whose vertices are all on $S^{n-1}_\infty$. A facet of a polyhedron $P$ is
the intersection of \( P \) with a codimension one hyperplane \( T \) such that exactly one component of \( \mathbb{H}^n \backslash T \) is disjoint from \( P \).

Figure 10: An ideal tetrahedron. Picture credit: Vladimir Bulatov, Oregon State University

An ideal tetrahedron (see Figure 10) then is a tetrahedron with all of its vertices on the sphere at infinity. The following lemma is what will allow us to put a hyperbolic structure on \( M \backslash \{v\} \).

**Proposition 1** Let \( F_1, F_2, \) and \( F_3 \) be three facets of an ideal tetrahedron in \( \mathbb{H}^3 \). Let \( \beta_{12} \) be the interior angle between \( F_1 \) and \( F_2 \) and define \( \beta_{23}, \beta_{31} \) similarly. Then \( \beta_{12} + \beta_{23} + \beta_{31} = \pi \).

**Proof 1** Since these three facets have a point at infinity, and the unique geodesic in the upper half-space model that intersects \( \infty \) is a vertical line all three facets are euclidean planes. Then \( \beta_{12}, \beta_{23}, \) and \( \beta_{31} \) form the interior angles of a euclidean triangle.

Thus if our ideal tetrahedron is regular (if there exists a group of isometries taking any vertex to any other vertex) then each of the angles \( \beta_{ij} = \pi/3 \). Finally, by examining Figure 3 we see that each point \( x \) of \( M \) lying in a one cell the two cells of \( M \) are glued to \( x \) six times. Therefore the six interior angles around \( x \) sum to \( 2\pi \) and so every point of \( M \backslash \{v\} \) has a hyperbolic neighborhood which proves Theorem 2.

**References**


