On Poincaré’s Theorem for Fundamental Polygons
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1. Introduction

Poincaré’s classical theorem of fundamental polygons is a widely known, valuable tool that gives sufficient conditions for a (convex) hyperbolic polygon, equipped with so-called side-pairing transformations, to be a fundamental domain for a discrete subgroup of isometries. Poincaré first published the theorem in dimension two in 1882. In the past century, there have been several published proofs of this theorem (though many of them are questionably valid). It is the goal of this paper to present a proof of Poincaré’s Fundamental Polygon Theorem. We first examine the relevant hyperbolic geometry.

Our chief references are Charles Walkden’s Hyperbolic Geometry class notes [2], in which many of the definitions and results can be found, as well as Svetlana Katok’s *Fuchsian Groups* [1], in which the proof can be found.

In what follows, we will denote the hyperbolic upper-half space \( \mathcal{U} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \), and its boundary \( \partial \mathcal{U} = \{ z \in \mathbb{C} : \text{Im}(z) = 0 \cup \{ \infty \} \} \). Let \( z_1, \ldots, z_n \in \mathcal{U} \cup \partial \mathcal{U} \). A hyperbolic \( n \)-gon with vertices \( z_1, \ldots, z_n \) refers to the region of \( \mathcal{U} \) bounded by the geodesic segments

\[ [z_1, z_2], \ldots, [z_{n-1}, z_n], [z_n, z_1]. \]

Note that the definition of a hyperbolic polygon allows the possibility that the polygon has at least one boundary vertex, and the possibility that the polygon has an edge lying on the boundary. While Poincaré’s Theorem holds in both these situations, we will assume this does not happen, i.e., all edges of the hyperbolic polygon are arcs of geodesics and no vertex is on the boundary.

Before we are ready to state (and prove) Poincaré’s theorem, let us start with a few preliminaries.

2. Preliminaries

2.1. Fuchsian Groups

Recall that the collection of Möbius transformations of \( \mathcal{U} \), \( \text{Möb}(\mathcal{U}) \), forms a group under composition. Also recall that \( \text{Möb}(\mathcal{U}) = \text{Isom}(\mathcal{U}^\alpha) \), where \( \text{Isom}(\mathcal{U}^\alpha) \) refers to the group of isometries on \( \mathcal{U}^\alpha \). There are many subgroups of the group of isometries on \( \mathcal{U}^\alpha \), but, in this paper, we are interested in one class of subgroups; namely those that are discrete.

**Definition 2.1.** A *Fuchsian group* is a discrete subgroup of \( \text{Möb}(\mathcal{U}) = \text{Isom}(\mathcal{U}^\alpha) \).

**Example 2.1.**

1. Any finite subgroup of \( \text{Möb}(\mathcal{U}) \) is a Fuchsian group.
2. \( \text{PSL}(2, \mathbb{Z}) \) is a Fuchsian group.
We can equivalently define a Fuchsian group in terms of its action on \( \mathcal{U} \), but to do so, we need to first define the notion of a properly discontinuous action.

**Definition 2.2.** A group \( \Gamma \) of homeomorphisms on a metric space \( X \) acts properly discontinuously if, for any compact set \( K \subset X, \gamma(K) \cap K \neq \emptyset \) for only finitely many \( \gamma \in \Gamma \).

**Theorem 2.1.** Let \( \Gamma \) be a subgroup of \( \text{PSL}(2, \mathbb{R}) \). Then \( \Gamma \) is a Fuchsian group if and only if it acts properly discontinuously on \( \mathcal{U} \).

**Proof.** Suppose first that \( \Gamma \) is a Fuchsian group. Let \( z \in \mathcal{U} \) and let \( K \) be a compact subset of \( \mathcal{U} \). Citing Katok (specifically Lemma 2.2.4), for any such point \( z \), and \( K \), the set \( E = \{ \gamma \in \text{PSL}(2, \mathbb{R}) : \gamma(z) \in K \} \) is compact. Now a subset of a compact set can only have infinitely many elements if one of its points is an accumulation point. But a space is discrete if and only if there are no accumulation points. The intersection, then, of a discrete set with a compact set contains only finitely many points. Hence \( E \cap \Gamma \) is a finite set, which proves that \( \Gamma \) acts properly discontinuously on \( \mathcal{U} \).

To prove the converse, we argue by way of contraposition. Suppose \( \Gamma \) is not discrete. Then there is a sequence of distinct transformations \( \gamma_k \) such that \( \gamma_k \to \text{Id} \), i.e. \( \{ \gamma_k(s) \} \) is a sequence of distinct points that converges to \( s \). Hence for any closed hyperbolic disc \( B_\epsilon(s) \) we have \( \gamma(s) \cap B_\epsilon(s) \neq \emptyset \) for infinitely many \( \gamma \in \Gamma \). But then \( \gamma(s) \cap B_\epsilon(s) \subset \gamma(B_\epsilon(s)) \cap B_\epsilon(s) \neq \emptyset \) for infinitely many \( \gamma \in \Gamma \). Thus \( \Gamma \) does not act properly discontinuously.

**Remark 2.1.** The same statements hold in the Poincaré ball model \( \mathbb{B} \).

### 2.2. Fundamental Domains

**Definition 2.3.** Let \( \Gamma \) be a Fuchsian group. A fundamental domain \( F \) for \( \Gamma \) is an open subset of \( \mathcal{U} \) such that

(i) \( \bigcup_{\gamma \in \Gamma} \gamma(F) = \mathcal{U} \).

(ii) The images \( \gamma(F) \) are pairwise disjoint; i.e. \( \gamma_1(F) \cap \gamma_2(F) = \emptyset \) for all \( \gamma_1, \gamma \in \Gamma \) with \( \gamma_1 \neq \gamma_2 \).

If \( F \) is a fundamental domain for \( \Gamma \), we say that the images of \( F \) under \( \Gamma \) tessellate \( \mathcal{U} \).

### 2.3. Side-Pairing Transformations

Let \( D \) be a hyperbolic polygon. A side \( s \subset \mathcal{U} \) of \( D \) is an edge of \( D \) in \( \mathcal{U} \) equipped with an orientation, i.e., an edge which starts at one vertex and ends at another.

Suppose \( \Gamma \) is a Fuchsian group. Let \( D \) be a hyperbolic polygon with \( s \subset \mathcal{U} \) a side of \( D \). Suppose that for some \( \gamma \in \Gamma \setminus \{ \text{Id} \} \), we have that \( \gamma(s) \) is also a side of \( D \).

**Definition 2.4.** We say that the sides \( s \) and \( \gamma(s) \) are paired and we call \( \gamma \) a side-pairing transformation.

**Remark 2.2.** \( \gamma^{-1} \in \Gamma \setminus \{ \text{Id} \} \) maps the side \( \gamma(s) \) back to the side \( s \).

Let \( \Gamma \) be a Fuchsian group and let \( D \) be a hyperbolic polygon. The following diagram illustrates which sides of \( D \) are paired and how the side-pairing transformations act.
2.4. Elliptic Cycles

Note that, as in the figure above, two sides \( s \) and \( *s \) of \( D \) a common vertex. We follow the notation in Walkden’s notes [2] to facilitate cross-referencing with his work. Let the pair \((v,s)\) denote a vertex \( v \) of \( D \) and a corresponding side \( s \) of \( D \). The pair comprising of the same vertex \( v \) and the other side \(*s\) the shares \( v \) is denoted \(* (v, s)\).

The following inductive process outlines how to calculate so-called elliptic cycles and their corresponding transformations:

(i) Let \( v = v_0 \) be a chosen vertex of \( D \) and let \( s_0 \) be a side with an end point \( v_0 \). Let \( \gamma_1 \) be the side-pairing transformation associated to the side \( s_0 \), so that \( \gamma_1 \) maps \( s_0 \) to, say, \( s_1 \).

(ii) Let \( s_1 = \gamma_1(s_0) \) and let \( v_1 = \gamma_1(v_0) \). This gives a new pair \((v_1, s_1)\).

(iii) Let \( \gamma_2 \) be the side-pairing transformation associated to the side \(*s_1\) (the side of \( D \) that shares with \( s_1 \) the end point \( v_1 \)). Then \( \gamma_2(*s_1) = s_2 \) and \( \gamma_2(v_1) = v_2 \), a vertex of \( D \).

(iv) Repeat (i) - (iii) inductively.

With this process, we obtain a sequence of pairs of vertices and sides:

\[
(v_0, s_0) \xrightarrow{\gamma_1} (v_1, s_1) \xrightarrow{*} (v_1, *s_1) \xrightarrow{\gamma_2} (v_2, s_2) \xrightarrow{*} \cdots \xrightarrow{\gamma_i} (v_i, s_i) \xrightarrow{*} (v_i, *s_i) \xrightarrow{\gamma_{i+1}} (v_{i+1}, s_{i+1}) \xrightarrow{*} \cdots
\]

As there are only finitely many pairs \((v, s)\), this process must eventually return to the initial pair \((v_0, s_0)\).

Let \( n \) be the least positive integer such that \((v_n, *s_n) = (v_0, s_0)\).

**Definition 2.5.** The sequence of vertices \( E = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1} \) is called an *elliptic cycle*. The transformation \( \gamma_n \gamma_{n-1} \cdots \gamma_2 \gamma_1 \) is called an *elliptic cycle transformation*.

This process is best understood via an example.

**Example 2.2.** Consider the polygon in the following figure.
Following the outlined process, we obtain the following sequence of pairs of vertices and sides:

\[
\begin{align*}
(A, s_1) & \xrightarrow{\gamma_1} (F, s_5) \xrightarrow{\ast} (F, s_6) \\
& \xrightarrow{\gamma_2} (E, s_4) \xrightarrow{\ast} (E, s_5) \\
& \xrightarrow{\gamma^{-1}_1} (B, s_1) \xrightarrow{\ast} (B, s_2) \\
& \xrightarrow{\gamma^{-1}_3} (D, s_3) \xrightarrow{\ast} (D, s_4) \\
& \xrightarrow{\gamma^{-1}_2} (A, s_6) \xrightarrow{\ast} (A, s_1).
\end{align*}
\]

Thus we have the elliptic cycle \( A \to F \to E \to B \to D \) with the corresponding elliptic cycle transformation \( \gamma^{-1}_2 \gamma^{-1}_3 \gamma^{-1}_1 \gamma_1 \). The vertex \( C \) is its own elliptic cycle with corresponding elliptic cycle transformation \( \gamma_3 \).

**Definition 2.6.** If an elliptic cycle transformation is the identity then we call the elliptic cycle an **accidental cycle**.

**Proposition 2.1.** Let \( \Gamma \) be a Fuchsian group with \( D \) a fundamental domain. Suppose \( D \) is a hyperbolic polygon with vertices \( v_1, \ldots, v_n \) in \( \mathcal{U} \) and let \( \mathcal{E} \) be an elliptic cycle. Then there exists a positive integer \( m_\mathcal{E} \) such that

\[
\sum_{i=1}^{n} \theta_{v_i} = \frac{2\pi}{m_\mathcal{E}},
\]

where \( \theta_{v_i} \) is the interior angle of \( D \) at the vertex \( v_i \).

Let \( D \) be a hyperbolic polygon with vertices \( v_1, \ldots, v_n \). Let \( \mathcal{E} \) be the elliptic cycle \( \mathcal{E} = v_0 \to v_1 \to \cdots \to v_{n-1} \).

We are particularly interested in elliptic cycles that satisfy the following condition:

*There exists a positive integer \( m_\mathcal{E} \) such that \( \sum_{i=0}^{n-1} \theta_{v_i} = \frac{2\pi}{m_\mathcal{E}}, \) where \( \theta_{v_i} \) is the interior angle of \( D \) at the vertex \( v_i \).*

**Definition 2.7.** Define condition \((*)\) to be the **elliptic cycle condition**. We shall call an elliptic cycle satisfying \((*)\) a **proper elliptic cycle**. We shall call \( m_\mathcal{E} \) the **cycle constant** corresponding to the proper elliptic cycle \( \mathcal{E} \). We note that this is absolutely not standard terminology; there does not seem to be any accepted names for elliptic cycles satisfying \((*)\) or for the constant \( m_\mathcal{E} \).

**Remark 2.3.** Recall that an accidental elliptic cycle has elliptic cycle transformation the identity. Hence the interior angle sum of an accidental elliptical cycle is \( 2\pi \).

### 3. Poincaré’s Theorem

We are now ready for the precise statement of Poincaré’s theorem. Using our new vocabulary, it states that, given a (convex) hyperbolic polygon, equipped with side-pairing transformations such that every elliptic cycle is a proper elliptic cycle, then the side-pairing transformations generate a Fuchsian group for which the hyperbolic polygon is a fundamental domain. Moreover, the theorem gives a presentation for this discrete subgroup. We present two versions.
**Theorem 3.1** (Poincaré’s Theorem, Version 1). Let $D$ be a convex hyperbolic polygon with finitely many sides. Suppose that all vertices lie inside $U$ and that $D$ is equipped with a collection side-pairing Möbius transformations. Suppose that no side of $D$ is paired with itself. Suppose that the elliptic cycles are $E_1, \ldots, E_r$, and that each $E_j$ is a proper elliptic cycle with corresponding cycle constant $m_j$. Then:

(i) The subgroup $\Gamma$ generated by all side-pairing transformations is a Fuchsian group.

(ii) $D$ is a fundamental domain for $\Gamma$.

(iii) $\Gamma = \langle \gamma_s \in G : \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = e \rangle$, where $\gamma_j = \gamma_{v,s}$ is an elliptic cycle transformation associated with $(v,s)$ (for some side $s$ with vertex $v$) corresponding to each elliptic cycle $E_j$.

**Remark 3.1.** The relations in (iii) appear the depend on which pair $(v, s)$ on the elliptic cycle $E_j$ is used to define $\gamma_j$. However, the relation $\gamma_j^{m_j}$ is independent of the choice of $(v, s)$. This follows from the fact that if $v'$ is any other vertex on the same elliptic cycle then $\gamma_{v', s'}$ is conjugate to either $\gamma_{v,s}$ or $\gamma_{v,s}^{-1}$.

**Theorem 3.2** (Poincaré’s Theorem, Version 2). Let $g \geq 0, r \geq 0, m_i \geq 2$, for $1 \leq i \leq r$ be integers such that

$$(2g - 2) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) > 0.$$  

Then there exists a Fuchsian group generated by the side-pairings of a hypobolic $n$-gon ($n$ dependent only on $g$ and $r$) with $r$ proper elliptic cycles. Moreover, each proper elliptic cycle has associated cycle constant $m_i$, $i = 1, \ldots, r$.

The following lemma proves that these two versions of Poincaré’s Theorem yield equivalent results. It will also serve as much of the proof of the second version of Poincaré’s Theorem (we only prove the second version).

**Lemma 3.1.**

(a) Let $D$ be a (convex) hyperbolic polygon, equipped with a collection of side-pairing transformations. Suppose that no side of $D$ is paired with itself. Suppose that the elliptic cycles are $E_1, \ldots, E_r$, and that each $E_j$ is a proper elliptic cycle with corresponding cycle constant $m_j$. Then

$$(2g - 2) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) > 0,$$

where $g$ is defined to be the genus of $D$.

(b) Conversely, given integers $g \geq 0, r \geq 0$, and $m_j \geq 2$, for $1 \leq j \leq r$. Suppose that

$$(2g - 2) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) > 0,$$

Then there exists a (convex) hyperbolic polygon, equipped with side-pairing transformations, that has proper elliptic cycles $E_1, \ldots, E_r$ such that $m_j$ is the cycle constant corresponding to $E_j$.

**Proof.**

$^1$The genus $g$ of a surface $X$ is given by $\chi(X) = 2 - 2g$, where $\chi(X) = V - E + F$ is the Euler characteristic.
(a) Suppose that the (convex) hyperbolic polygon $D$ has $n$ vertices (and hence $n$ sides). Let $E_1, \ldots, E_r$ be the non-accidental elliptic cycles. From Proposition 2.1, the angle sum along the elliptic cycle $E_j$ is

$$\sum_{i=1}^{n} \theta_{v_i} = \frac{2\pi}{m_j},$$

where $\theta_{v_j}$ is the interior angle of $D$ at the vertex $v_j$ in $E_j$. Suppose there are $s$ accidental cycles. From Proposition 2.1, the angle sum along an accidental cycle is $2\pi$, and thus the internal angle sum along all accidental cycles is $2\pi s$. Therefore, the internal angle sum of $D$ is given by

$$2\pi \left( \sum_{i=1}^{r} \frac{1}{m_i} + s \right).$$

By the Gauss-Bonnet formula for the area of a hyperbolic polygon, we have

$$\text{Area}_{\mathcal{U}}(D) = (n - 2)\pi - 2\pi \left( \sum_{i=1}^{r} \frac{1}{m_i} + s \right).$$

Now, under the side-pairing transformations, we have $V = r + s$ vertices, as each elliptic cycle gives one vertex, $E = n/2$ edges, as no side is paired with itself, and $F = 1$ face. Hence

$$2 - 2g = \chi(D) = V - E + F = r + s - \frac{n}{2} + 1.$$ 

And therefore,

$$\text{Area}_{\mathcal{U}}(D) = 2\pi \left( r + s - (2 - 2g) - \sum_{i=1}^{r} \frac{1}{m_i} - s \right) = 2\pi \left( (2g - 2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right).$$

As this quantity must be positive, the condition

$$(2g - 2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0$$

is satisfied.

(b) Choose integers $g \geq 0, r \geq 0,$ and $m_j \geq 2,$ for $1 \leq j \leq r$ satisfying $(2g - 2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0$. We shall use the unit ball model $B$ of hyperbolic geometry. The following figures illustrate this proof with $g = 2, r = 4, m_i > 2$ for $i = 1, 2, 3$, and $m_4 = 2$.

Begin by drawing $4g + r$ radii from the center of $B$ so that the angle between any two adjacent radii is $\theta = 2\pi/(4g + r)$. On each radius, choose a point (Euclidean) distance $t$, for $0 < t < 1$, from the origin and join adjacent points with a hyperbolic geodesic. This gives a hyperbolic polygon $M(t)$ with $4g + r$ vertices. Starting at an arbitrary point, label the vertices clockwise $v_1, \ldots, v_r, v_{1,1}, \ldots, v_{1,4}, v_{2,1}, \ldots, v_{2,4}, \ldots, v_{g,1}, \ldots, v_{g,4}$.\footnote{If $P$ is a hyperbolic $n$-gon with vertices $v_1, \ldots, v_n$ and internal angles $\theta_1, \ldots, \theta_n$ then $\text{Area}_{\mathcal{U}}(P) = (n - 2)\pi - \sum_{i=1}^{n} \theta_i$.}
On each of the first $r$ sides, construct an isosceles hyperbolic triangle, with the $M(t)$ side as the base and the vertex external to $M(t)$. Label the external vertices $w_1, \ldots, w_r$. Note that the internal angle at $w_i$ is $2\pi/m_i$. When $m_i = 2$, then $2\pi/m_i = \pi$ and we have a degenerate triangle, i.e. $w_i$ is just the midpoint of the side of $M(t)$.

The union of these triangles with $M(t)$ forms a new hyperbolic polygon $N(t)$ with $4g + 2r$ sides. For $\ell = 1, \ldots, g$ and $j = 1, \ldots, r$, label the sides of $N(t)$ by $s(v_\ell), s(v_{\ell,j}), s(w_j)$, where the side $s(v)$ is immediately clockwise of vertex $v$. Orient each side as in the figure below.

For each pair of geodesics in $B$ there exists an orientation-preserving isometry of $B$ that maps one to the other. Let $\gamma_j$, for $1 \leq j \leq r$ be the side-pairing transformation pairing $s(v_j)$ with $s(w_j)$. Note
that $\gamma_j$ is just a rotation about $w_j$ through the angle $2\pi/m_j$. Let $\gamma_{\ell,1}$ and $\gamma_{\ell,2}$, for $1 \leq \ell \leq g$, be the side-pairing transformation pairing $s(v_{\ell,3})$ with $s(v_{\ell,1})$ and $s(v_{\ell,2})$ with $s(v_{\ell,4})$, respectively.

Note that as $t \rightarrow 0$, so does $\text{Area}_{U}(N(t))$. But also, using again the Gauss-Bonnet formula for a hyperbolic $n$-gon (and the fact that the interior angle of any vertex at infinity is zero), we have that, as $t \rightarrow 1$,

$$\text{Area}_{U}(N(t)) \rightarrow (4g + 2r - 2)\pi - 2\pi \left( \sum_{i=1}^{r} \frac{1}{m_i} \right) = 2\pi \left( (2g - 1) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).$$

Since the hyperbolic area $\text{Area}_{U}(N(t))$ is certainly a continuous function of $t$, the Intermediate Value Theorem guarantees the existence of $t_0 \in (0, 1)$ such that the hyperbolic area of $N(t_0)$ is exactly

$$\text{Area}_{U}(N(t_0)) = 2\pi \left( (2g - 1) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).$$

We now calculate the elliptic cycles and show that each elliptic cycle is proper. First we have the elliptic cycle

$$v_{1,1} \rightarrow v_{1,4} \rightarrow v_{1,3} \rightarrow v_{1,2} \rightarrow v_{2,1} \rightarrow v_{2,4} \rightarrow v_{2,3} \rightarrow v_{2,2} \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4,$$

with corresponding elliptic cycle transformation

$$\gamma_4 \gamma_3 \gamma_2 \gamma_1 \gamma_{2,1} \gamma_{2,2}^{-1} \gamma_{1,2}^{-1} \gamma_{1,1} \gamma_{1,2}^{-1} \gamma_{1,1}^{-1}.$$  

And elliptic cycles $w_1, w_2, w_3, w_4$ with elliptic cycle transformations $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, respectively. Note that, for $j = 1, \ldots, r$, the internal angle sum is given only by the internal angle at $w_j$, which is, by construction, $2\pi/m_j$. Hence each elliptic cycle is a proper elliptic cycle with cycle constant $m_j$.  


3.1. Proof of Poincaré’s Theorem, Version 2

With this tool in hand (Lemma 3.1), proving the Poincaré Theorem (the second version, Theorem 3.2) amounts only to showing that the side-pairings $\gamma_j, \gamma_{t,1}, \gamma_{t,2}$, for $1 \leq j \leq r$ and $1 \leq \ell \leq g$, generate a Fuchsian group for which the hyperbolic $(4g + 2r)$-gon, $N(t_0)$, constructed in the previous section is a fundamental domain.

**Proof (sketch).** Let $\Gamma$ be the group generated by

$$\{\gamma_j, \gamma_{t,1}, \gamma_{t,2} : 1 \leq j \leq r \text{ and } 1 \leq \ell \leq g\}.$$

It can be shown that $N(t_0)$ tessellates $\mathcal{B}$ via the transformations in $\Gamma$, and hence is a fundamental domain for $\Gamma$. A compact (hence closed and bounded) region in $\mathcal{B}$ intersects with only finitely many $\Gamma$-images of $N(t_0)$, and so we see that $\Gamma$ acts properly discontinuously on $\mathcal{B}$. Hence, using Theorem 2.1, $\Gamma$ is indeed a Fuchsian group.

3.2. Using Poincaré’s Theorem

**Example 3.1 (A Hyperbolic Octagon).** Let $P$ be a regular hyperbolic octagon (with each internal angle equal to $\pi/4$). As in the following figure, label the vertices of $P$ $v_1, \ldots, v_8$ counter-clockwise and label the sides $s_1, \ldots, s_8$ so that $s_i$ occurs immediately after vertex $v_i$. 

![Hyperbolic Octagon Diagram](image-url)
We calculate the elliptic cycles:

\[
\begin{align*}
(v_1, s_1) & \xrightarrow{\gamma_1} (v_4, s_3) \xrightarrow{\star} (v_4, s_4) \\
& \xrightarrow{\gamma_2} (v_3, s_2) \xrightarrow{\star} (v_3, s_3) \\
& \xrightarrow{\gamma_3^{-1}} (v_2, s_1) \xrightarrow{\star} (v_2, s_2) \\
& \xrightarrow{\gamma_2^{-1}} (v_5, s_5) \xrightarrow{\star} (v_5, s_5) \\
& \xrightarrow{\gamma_3} (v_8, s_7) \xrightarrow{\star} (v_8, s_8) \\
& \xrightarrow{\gamma_4} (v_7, s_6) \xrightarrow{\star} (v_7, s_7) \\
& \xrightarrow{\gamma_3^{-1}} (v_6, s_5) \xrightarrow{\star} (v_6, s_6) \\
& \xrightarrow{\gamma_4^{-1}} (v_1, s_8) \xrightarrow{\star} (v_1, s_1).
\end{align*}
\]

Thus, there is just one elliptic cycle:

\[
\mathcal{E} = v_1 \rightarrow v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_5 \rightarrow v_8 \rightarrow v_7 \rightarrow v_6,
\]

with corresponding elliptic cycle transformation:

\[
\gamma_4^{-1} \gamma_3^{-1} \gamma_4 \gamma_3 \gamma_2^{-1} \gamma_1^{-1} \gamma_2 \gamma_1.
\]

As each interior angle is \(\pi/4\), and since \(8 \times \frac{\pi}{4} = 2\pi\), this elliptic cycle is proper (with cycle constant 1). Thus by Poincaré’s Theorem (the first version) the group generated by the side-pairing transformations \(\gamma_1, \gamma_2, \gamma_3, \gamma_4\) generate a Fuchsian group, \(\Gamma\). Moreover, we can write

\[
\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_4^{-1} \gamma_3^{-1} \gamma_4 \gamma_3 \gamma_2^{-1} \gamma_1^{-1} \gamma_2 \gamma_1 = e \rangle.
\]

4. References


2. C. Walkden; Hyperbolic Geometry. Lecture notes (Manchester 2016). Available at: [http://www.maths.manchester.ac.uk/~cwalkden](http://www.maths.manchester.ac.uk/~cwalkden)