The Geometrization Theorem

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The geometrization theorem, originally conjectured by William Thurston in 1982, asserts that any compact, orientable 3-manifold can be broken into finitely many pieces in such a way that each piece admits exactly one of a list of eight model geometries. This is analogous to Thurston’s uniformization theorem for surfaces, which states that any simply connected 2-manifold is conformally equivalent to either the plane, the sphere, or the disk. There is no known analogous result for higher dimensions. Indeed, the techniques used to prove the Geometrization theorem have not yet been extended to dimensions 4 and above.

In this paper, we will discuss some ingredients for the proof of geometrization including a basic description of the canonical decomposition. Later, we will mention several properties of the eight model geometries for 3-manifolds.

1 Statement of the theorem

**Theorem 1** (Geometrization). Every compact, orientable 3-manifold $M$ admits a canonical decomposition into finitely many pieces such that each piece admits exactly one of the following geometries:

1. Euclidean geometry
2. Hyperbolic geometry
3. Spherical geometry
4. The geometry of $S^2 \times \mathbb{R}$
5. The geometry of $H^2 \times \mathbb{R}$
6. The geometry of $\widetilde{SL(2, \mathbb{R})}$
7. Nil geometry
8. Sol geometry

1.1 Locally homogeneous Riemannian metrics

In order to gain some intuition about these geometries, we may regard the geometric structure on each piece from the decomposition of $M$ as a complete, locally homogeneous Riemannian metric. Recall several definitions:

**Definition 1.** We say a metric $g$ on $M$ is homogeneous if, for all $x, y \in M$, there exists an isometry $\varphi : M \to M$ with respect to $g$ such that $\varphi(x) = y$. 
We say that \( g \) is **locally homogeneous** if, for all \( x, y \in M \), there exist neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively and an isometry \( \varphi : U \to V \) with respect to \( g \) such that \( \varphi(x) = y \).

The following result shows that on a simply connected manifold, these two definitions are equivalent.

**Proposition 1 ([11]).** Any locally homogeneous metric \( g \) on a simply connected manifold is homogeneous.

This, combined with the fact that any homogeneous metric is complete, tells us that if \((M, g)\) is locally homogeneous, then its universal cover \( \tilde{M} \) admits a complete, homogeneous metric \( \tilde{g} \) (obtained via lifting \( g \)).

Thus, if we let \( G = \text{Isom}(\tilde{M}) \), then \( M \) admits a \((G, \tilde{M})\)-structure, and \( g \) coincides with the metric induced by \( \tilde{g} \).

### 1.2 The decomposition

Here, we will describe the ‘canonical decomposition’ used in the Geometrization theorem. We will make use of several definitions.

**Definition 2.** An orientable closed 3-manifold \( M \) is **prime** if it is not diffeomorphic to \( S^3 \) and in any connected sum \( M = M_1 \# M_2 \), either \( M_1 \) or \( M_2 \) is diffeomorphic to \( S^3 \).

**Definition 3.** An orientable closed 3-manifold \( M \) is **irreducible** if every separating embedded 2-sphere bounds a 3-ball.

The motivation for these definitions comes from the idea of an **essential** 2-sphere, i.e. a sphere which does not bound a 3-ball. If such a sphere is separating (disconnects the manifold), then it gives rise to a connected sum decomposition. If it is not separating, then one can obtain a decomposition of the form \( M = (S^2 \times S^1) \# M' \) (see [2]). One can repeat this process for as long as some manifold in the connected sum has an essential 2-sphere. The fact that this process terminates, and the uniqueness of the resulting decomposition, is shown in the following theorem.

**Theorem 2** (Prime decomposition theorem, [6], [7]). Every compact, orientable 3-manifold is the connected sum of a finite collection of prime manifolds, each of which is either irreducible or \( S^2 \times S^1 \). The collection is unique up to homeomorphism and re-ordering.

This reduces the problem to the study of compact, orientable, irreducible 3-manifolds (see [1]). These can then be decomposed further, but to do so we will need several more definitions.

**Definition 4.** Let \( \Sigma^2 \subset M^3 \) be a compact, connected, properly embedded surface. \( \Sigma \) is said to be **incompressible** if one of the following hold:

1. \( \Sigma \neq S^2 \) or \( D^2 \) and the inclusion map induces an injective homomorphism in the fundamental group;
2. \( F = S^2 \) is an essential 2-sphere;
3. \( \Sigma = D^2 \) and \( \partial \Sigma \) is not null homotopic in \( \partial M \).
Definition 5. An irreducible manifold with boundary $M^3$ is said to be geometrically atoroidal if every incompressible torus $T^2 \subset M^3$ is isotopic to a component of $\partial M$.

A closed manifold is geometrically atoroidal if it has no incompressible tori.

Definition 6. A compact manifold $M^3$ is said to be a Siefert fiber space if it admits a foliation by $S^1$ fibers.

Though this is not the original definition, this has since been shown to be equivalent. The Siefert fiber spaces have been classified.

The following theorem then completes the decomposition.

Theorem 3 (Torus decomposition theorem, [4], [5]). Let $M^3$ be compact, orientable, and irreducible. Then there exists a (possibly empty) finite collection of disjoint incompressible tori $\{T^2_i\}$ such that each component of $M \setminus \bigcup T^2_i$ is either geometrically atoroidal or a Siefert fiber space. A minimal such collection is unique up to homotopy.

1.3 Some notes on the proof

The proof of the Geometrization theorem was completed by Grisha Perelman in 2003. Prior to his work, the theorem was only known to hold under particular assumptions.

Thurston’s original work on geometrization handled the Siefert pieces and the geometrically atoroidal pieces with nonempty boundary. He proved that the interior of such an atoroidal piece admits a finite volume, complete hyperbolic metric. He also proved that the interior of any compact Siefert space admits a complete, locally homogeneous metric of finite volume. This reduces the problem to proving geometrization in the following cases. Suppose $M^3$ is closed, orientable, irreducible, and has no incompressible tori. Then one of the following holds:

1. $\pi_1(M)$ is infinite and has no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$
2. $\pi_1(M)$ has a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$
3. $\pi_1(M)$ is finite.

It was conjectured that in the first case $M$ should admit a hyperbolic metric, in the second $M$ should be a Siefert space, and in the third that it should admit a spherical metric. It was shown by the work of Thurston and many others (see [2], chapter 9) that this holds in the case of a Haken manifold (i.e., one which admits a two-sided incompressible surface), and (2) was shown to hold in general. However, cases (1) and (3) (hyperbolization and elliptization, respectively) remained open until the work of Perelman.

Perelman used techniques of geometric analysis to settle these remaining cases (see [8], [9], and [10]). His proof makes use of a program of Ricci flow with surgery, an initial prototype for which was introduced by Richard Hamilton in [3]. This is needed to account for the possibility of a flow where the curvature goes to infinity in finite time in some regions while remaining bounded in others. Such a situation cannot be remedied by rescaling space and time, and once again the manifold must be split into pieces which are analyzed separately.
2 The model geometries

Euclidean geometry: Euclidean geometry is modeled on $E^3$, i.e. $\mathbb{R}^3$ with the flat metric. Its isometry group is generated by orthogonal transformations and translations. There are ten compact manifolds which admit this geometry.

Hyperbolic geometry: Hyperbolic geometry is modeled on $H^3$, the space of constant negative curvature. Its isometry group is generated by reflections.

Spherical geometry: Spherical geometry is modeled on the sphere $S^3$, i.e. the space of constant positive curvature. $S^3$ can be embedded in $\mathbb{R}^4$ in such a way that its metric is induced by the flat metric on $\mathbb{R}^4$, where it can be shown that the full isometry group of $S^3$ is $O(4)$. The only subgroups of $O(4)$ which act transitively and discretely on $S^3$ have finite order, and thus a manifold with spherical geometry has a finite fundamental group.

$S^2 \times \mathbb{R}$: The isometry group of $S^2 \times \mathbb{R}$ is the Cartesian product $\text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$, where $\text{Isom}(S^2)$ can be identified with $O(3)$ (via embedding in $\mathbb{R}^3$) and $\text{Isom}(\mathbb{R})$ is generated by translations and reflections. There are only 4 compact manifolds which admit this geometry. An example is $\mathbb{R}P^2 \times S^1$, which is the quotient of $S^2 \times \mathbb{R}$ by $\langle (\text{Id}_+, T), (A, \text{Id}_+) \rangle$, where $T$ is a translation and $A$ is the antipodal map.

$H^2 \times \mathbb{R}$: As in the previous case, the isometry group of $H^2 \times \mathbb{R}$ is given by the product $\text{Isom}(H^2) \times \text{Isom}(\mathbb{R})$. As is clear from our discussion in class, there are infinitely many manifolds which admit this geometry (for example, the product of any hyperbolic 2-manifold with the circle).

$\widetilde{SL}(2, \mathbb{R})$: $\widetilde{SL}(2, \mathbb{R})$ is the universal cover of $SL(2, \mathbb{R})$. To obtain the metric on $\widetilde{SL}(2, \mathbb{R})$, we first look at 2-dimensional hyperbolic space $H^2$. Its metric induces a metric on its unit tangent bundle $UH^2$. $UH^2$ can then be identified with $PSL(2, \mathbb{R})$, which also has universal cover $\widetilde{SL}(2, \mathbb{R})$.

Nil geometry: Nil geometry is modeled on the geometry of the nilpotent group generated by $3 \times 3$ matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$ under multiplication. This group is also called the Heisenberg group. It can be identified with $\mathbb{R}^2$ via

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z),$$

and in this coordinate system the metric is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$ 

An example of a manifold with Nil geometry is the quotient of Nil by its matrices with integer entries.

Sol geometry: Sol geometry is modeled on the solvable group $\mathbb{R}^2 \ltimes \mathbb{R}$ with the metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$ 

For a thorough description of these eight geometries, see [12].
References


