The Ping-Pong Lemma

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The Ping-Pong Lemma is a useful tool for determining when a group acting on a set is free or contains a free subgroup. This paper will focus on the proof of the Ping-Pong Lemma and examples of its use.

1 The Ping-Pong Lemma

Theorem. *(Ping-pong lemma)* Let G be a group acting on a set X. Let Γ_1 and Γ_2 be two subgroups of G and let Γ be the subgroup generated by Γ_1 and Γ_2. Assume that Γ_1 contains at least 3 elements and Γ_2 contains at least 2 elements. Assume also that there exist two nonempty subsets X_1 and X_2 of X with X_2 not contained in X_1 such that

$$\gamma(X_2) \subset X_1 \text{ for all } \gamma \in \Gamma_1, \gamma \neq \text{Id}.$$

$$\gamma(X_1) \subset X_2 \text{ for all } \gamma \in \Gamma_2, \gamma \neq \text{Id}.$$

Then Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$.

Proof. The following proof appears in [1].

It suffices to show that any word in Γ spelled from letters in the disjoint union of $\Gamma_1 \setminus \{\text{Id.}\}$ and $\Gamma_2 \setminus \{\text{Id.}\}$ is not the identity. Let w be such a word.

Let a_i be elements of Γ_1 and b_i be elements of Γ_2. We can now split the possibilities for w into four cases.

Case 1. Suppose $w = a_1 b_1 ... b_{k-1} a_k$. Then

$$\begin{aligned} w(X_2) = a_1 b_1 ... b_{k-1} a_k(X_2) &\subset a_1 b_1 ... a_{k-1} b_{k-1}(X_1) \\ &\subset a_1 b_1 ... a_{k-1}(X_2) \\ &\vdots \\ &\subset a_1(X_2) \\ &\subset X_1 \end{aligned}$$

Since $X_2 \not\subset X_1$, this shows that w is not the identity.

Case 2. Suppose $w = b_1a_1...a_{k-1}b_k$. Chooose any $a \in \Gamma_1 \setminus \{\text{Id.}\}$. Then the argument from case 1 shows that awa^{-1} is not the identity, therefore w is not the identity.

Case 3. Suppose $w = a_1b_1...a_kb_k$. Choose $a \in \Gamma_1 \setminus \{\text{Id.}, a_1^{-1}\}$. The argument from case 1 shows that awa^{-1} is not the identity, therefore w is not the identity.

Case 4. Suppose $w = b_1a_1...b_ka_k$. Choose $a \in \Gamma_1 \setminus \{\text{Id.}, a_k\}$. Case 1 shows that awa^{-1} is not the identity, therefore w is not the identity. □

This version of the Ping-Pong Lemma shows that Γ is a free product. The hypotheses can be modified slightly to get a different statement of the theorem.

Theorem. *(Ping-pong Lemma, alternate)* Let $G = \langle a, b\rangle$ be a group generated by two elements where both a and b have infinite order, and let G act on a set X. Suppose that there exist nonempty subsets X_1 and X_2 of X with X_2 not included in X_1 such that for all $n \in \mathbb{Z} \setminus \{0\}$,

$$a^n \cdot X_2 \subset X_1, \quad b^n \cdot X_1 \subset X_2$$

Then G is freely generated by $\{a, b\}$.

For a proof of this version of the lemma, see [2].

2 Examples

The following examples illustrate some of the ways in which the Ping-Pong Lemma can be used.

Example 1. Let $k \geq 1$ be an integer, and let $D_1, ..., D_{2k}$ be a set of closed discs with disjoint interiors.

For each $j \in \{1, ..., k\}$, let γ_j be a Möbius transformation mapping the exterior of D_{2k} onto the interior of D_{2k-1}. Let Γ be the subgroup of $PSL(2,\mathbb{C})$ generated by the γ_j. Then Γ is free on the elements γ_j.

Proof. This statement can be proved by generalizing the Ping-Pong Lemma to subgroups of $PSL(2,\mathbb{C})$ generated by a finite number of generators. See [1].

Example 2. *(Free subgroups of $SL(2,\mathbb{Z})$)* Let $k \in \mathbb{Z}$, $k \geq 2$. The two matrices $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ genearate a free subgroup of rank 2 in $SL(2,\mathbb{Z})$

(where $SL(2,\mathbb{Z})$ acts linearly on $\mathbb{R}^2$ in the usual way).

Proof. Let

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & kn \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z}) : n \in \mathbb{Z} \right\}$$

$$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ kn & 1 \end{pmatrix} \in SL(2,\mathbb{Z}) : n \in \mathbb{Z} \right\}$$

be the subgroups generated by $\gamma_1 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ respectively. Define two subsets X_1, X_2 of $\mathbb{R}^2$ by

$$X_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| > |y| \right\}$$

$$X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |y| > |x| \right\}$$

Clearly X_2 is not contained in X_1. Let $\varphi_1 = \begin{pmatrix} 1 & n_1 k \\ 0 & 1 \end{pmatrix}$ be any element of Γ_1 that is not equal to the identity (so, $n_1 \neq 0$), and let $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be any element of X_2.Then

$$\varphi_1 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & n_1 k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 + k n_1 y_2 \\ y_2 \end{pmatrix}$$

Since $|y_2| > |x_2|$, we have

$$|x_2 + k n_1 y_2| > |(|kn_1| - 1)y_2| = (|kn_1| - 1)|y_2| \geq |y_2|$$

Therefore, $\varphi_1 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in X_1$. Similarly, let $\varphi_2 = \begin{pmatrix} 1 & 0 \\ kn_2 & 1 \end{pmatrix} \in \Gamma_2$ be any element that is not the identity and let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in X_2$. Then

$$\varphi_2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ kn_2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ kn_2 x_1 + y_1 \end{pmatrix}$$

and because $|x_1| > |y_1|$,

$$|kn_2 x_1 + y_1| > |(|kn_2| - 1)x_1| \geq |x_1|$$

Therefore $\varphi_2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in X_2$. We can now apply the Ping-Pong Lemma to say that the group generated by γ_1 and γ_2 is a free subgroup of $SL(2,\mathbb{Z})$ and is isomorphic to $\Gamma_1 * \Gamma_2$. □

We can see explicitly why this does not work for $k = 1$, because the element

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the subgroup generated by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ has order 4.

Example 3. Let q be an integer greater than or equal to 3 and let $\lambda = 2\cos(\frac{\pi}{q})$. Consider the elements

$$a_\lambda = \begin{bmatrix}1 & \lambda\\ 0 & 1\end{bmatrix} \text{ and } j = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

in $G = PSL(2,\mathbb{R})$ and the subgroup Γ_λ of G generated by them. Then

$$\Gamma_\lambda \cong (\mathbb{Z}/q\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$$

Proof. For any $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in PSL(2,\mathbb{C})$ acts on z as

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} z = \frac{az+b}{cz+d}$$

and, when $c \neq 0$,

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \infty = \frac{a}{c} \quad \begin{bmatrix}a & b\\ c & d\end{bmatrix} \frac{-d}{c} = \infty$$

So, it is easy to see that

$$a_\lambda(z) = z + \lambda, \quad j(z) = -\frac{1}{z}$$

for all $z \in \hat{\mathbb{C}}$.

Consider the element $b_\lambda := a_\lambda j$ and the subgroup generated by it, $\Gamma_1 = \langle b_\lambda \rangle$. We can see that

$$b_\lambda(z) = a_\lambda\left(-\frac{1}{z}\right) = -\frac{1}{z} + \lambda = \frac{-\lambda z + 1}{-z}$$

So, b_λ is represented by the matrix $\begin{bmatrix}-\lambda & 1\\ -1 & 0\end{bmatrix}$. The eigenvalues of this matrix are $-e^{\pm i\pi/q}$, so the matrix can be diagonalized as

$$\begin{bmatrix}-e^{-i\pi/q} & e^{i\pi/q}\\ -1 & 1\end{bmatrix} \begin{bmatrix}-\lambda & 1\\ -1 & 0\end{bmatrix} \begin{bmatrix}-e^{-i\pi/q} & e^{i\pi/q}\\ -1 & 1\end{bmatrix} = \begin{bmatrix}-e^{-i\pi/q} & 0\\ 0 & -e^{i\pi/q}\end{bmatrix}$$

This shows that b_λ is conjugate to a rotation of order q, which, when taken to the qth power, is a scalar multiple of the identity matrix. So, $\Gamma_1 \cong \mathbb{Z}/q\mathbb{Z}$.

Let $\Gamma_2 = \langle j \rangle = \{\text{Id.}, j\}$. As j clearly has order 2, this group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Now consider the action of G only on $\hat{\mathbb{R}}$, and let

$$X_1 = (0, \infty], \quad X_2 = (-\infty, 0]$$

Clearly, $j(X_1) \subset X_2$.

We can see that $b_\lambda(0) = \infty$, and for $z \in X_2 \setminus \{0\}$, $b_\lambda(z) = \lambda - \frac{1}{z} \in (\lambda, \infty)$. So, $b_\lambda(X_2) \subset (\lambda, \infty]$.

Note that $b_\lambda(\infty) = \lambda$. For $z \in [\lambda, \infty)$, $b_\lambda(z) \in [\lambda - \frac{1}{\lambda}, \infty) \subset (0, \infty)$. Similarly, for $z \in [\lambda - \frac{1}{\lambda}, \infty)$, $b_\lambda(z) \in [\lambda - \frac{1}{\lambda - 1/\lambda}, \infty)$. We can continue in this fashion to show that $b_\lambda^j(-\infty, 0] \subset (0, \infty]$ for $j \in \{2, ..., q-1\}$. Therefore, every element of Γ_1 maps X_2 into X_1.

We may now apply the Ping Pong Lemma to conclude that $\Gamma_\lambda \cong \Gamma_1 * \Gamma_2 \cong (\mathbb{Z}/q\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. □

Example 4. *(Automorphisms of $\mathbb{R}$)* Let f be defined by

$$f(x) = \begin{cases} 4t & \text{if } t \in \left[0, \frac{1}{5}\right] \\ \frac{4}{5} + \frac{1}{4}\left(t - \frac{1}{5}\right) & \text{if } t \in \left[\frac{1}{5}, 1\right] \end{cases}$$

Let $[t]$ denote the integral part of a real number t, i.e. $[t]$ is the greatest integer less than or equal to t, and let $\{t\} = t - [t] \in [0, 1]$. Let $\gamma_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$\gamma_1(t) = [t] + f(\{t\})$$

Let T be the translation $t \mapsto t - \frac{1}{2}$, and define another function $\gamma_2 : \mathbb{R} \to \mathbb{R}$ by

$$\gamma_2 = T\gamma_1 T^{-1}$$

Then γ_1, γ_2 generate a free goup of rank 2 in the group $\text{Homeo}_+(\mathbb{R})$ of all orientation-preserving piecewise linear homeomorhpisms of the real line.

Proof. For $n \in \mathbb{Z}$, let $f^n = f \circ f \circ ... \circ f$ (n times) . Observe that

$$f^n\left(\left[\frac{1}{5}, 1\right]\right) \subset \left[\frac{4}{5}, 1\right] \qquad \text{for } n \geq 1$$

$$f^n\left(\left[0, \frac{4}{5}\right]\right) \subset \left[0, \frac{1}{5}\right] \qquad \text{for } n \leq -1$$

Define two subsets $X_1, X_2 \subset \mathbb{R}$:

$$X_1 = \bigcup_{k \in \mathbb{Z}} \left[k - \frac{1}{5}, k + \frac{1}{5}\right]$$

$$X_2 = \bigcup_{k \in \mathbb{Z}} \left[k + \frac{1}{2} - \frac{1}{5}, k + \frac{1}{2} + \frac{1}{5}\right] = T(X_1)$$

Then, because of what we've observed about f, $\gamma_1^n(X_2) \subset X_1$ for all integers $n \neq 0$. This shows that

$$\gamma_2^n(X_1) = T\gamma_1^n T^{-1}(T(X_2)) = T\gamma_1^n(X_2) \subset T(X_1) = X_2$$

for all integers $n \neq 0$.

We can now apply the Ping-Pong Lemma to prove the statement. □

References

[1] Pierre De La Harpe. *Topics in Geometric Group Theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago. ISBN 0-226-31719-6; Ch. II.B "The table-Tennis Lemma (Klein's criterion) and examples of free products"; pp. 2541.

[2] Clara Löh, *Geometric group theory, an introduction.* Retrieved from `http://www.mathematik.uni-regensburg.de/loeh/teaching/ggt_ws1011/lecture_notes.pdf`.