The Geometrization Theorem
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In this paper, we discuss the Geometrization Theorem, formerly Thurston’s Geometrization Conjecture, which is essentially the statement that one can cut up a 3-manifold into pieces such that each piece is geometrically “like” one of eight model geometries. The proof of this by Perelman in the early 2000s using the Ricci flow constitutes one of the most important results in mathematics in decades.

Our main reference is Scott [1], which is a nice introduction for those who wish to learn more about the subject. For the Ricci flow, Hamilton’s original paper [2] and the book by Chow and Knopf [3] are good introductions. Lee’s books [4] and [5] are good references for the basics of covering spaces, group actions, and Riemannian geometry. Special thanks to Professors Julien Paupert and Brett Kotschwar for helpful discussions.

1 2-Dimensional Geometries

In this section, we define the 2-dimensional model geometries and state the Uniformization Theorem as motivation for what is to come in three dimensions.

There are three model geometries in two dimensions. The simplest is the Euclidean space $E^2$, which is just $\mathbb{R}^2$ as a topological space endowed with the usual Euclidean metric. $E^2$ has curvature 0 everywhere. (Throughout, “curvature” will be taken to mean “Ricci scalar curvature”.)

Next, we have $S^2$, the 2-sphere, which inherits a metric from $E^3$, the 3-dimensional analogue of $E^2$. This metric has constant positive curvature equal to 1.

Finally, we have $H^2$, the Poincaré model of hyperbolic space. Topologically, $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the subspace topology. (In particular, $H^2$ is homeomorphic to $\mathbb{R}^2$.) We endow $H^2$ with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. With this metric, the geodesics in $H^2$ are vertical lines and arcs of circles which meet the $x$-axis orthogonally. With this metric, $H^2$ has constant negative curvature equal to $-1$.

Now, we have been referring to $E^2$, $S^2$, and $H^2$ as “model geometries” for a reason. We would like a notion under which a topological space admits a geometric structure like one of our three models. One such notion is encapsulated in the following definition:

**Definition.** Let $F$ be a topological space and $X$ be a Riemannian manifold. If there exists a group $\Gamma$ of isometries of $X$ such that $X \to X/\Gamma$ is a covering map and $F$ is homeomorphic to $X/\Gamma$, then we say that $F$ possesses a geometric structure modeled on $X$.

We have the following “big theorem” for 2-dimensional geometries:

**Theorem (Uniformization Theorem).** Every compact connected surface possesses a geometric structure modeled on one and only one of $S^2$, $E^2$, and $H^2$.

We can actually say more. The classification goes by Euler characteristic. Specifically, (compact, connected) surfaces of positive, zero, and negative Euler characteristic possess geometric structures modeled on the positive curvature model space $S^2$, the zero curvature model space $E^2$, and the negative curvature model space $H^2$, respectively. Indeed, the surfaces of positive Euler characteristic are $S^2$ (with $\chi = 2$) and the projective plane $P^2$ (with $\chi = 1$). Of course, $S^2$ has a geometric structure modeled on itself. As for $P^2$, the natural quotient map $S^2 \to P^2$ is a double-cover. Therefore, $P^2$ inherits a metric of curvature 1 from $S^2$.

The surfaces with Euler characteristic zero are the torus, $T^2$, and the Klein bottle, $K^2$. Both possess geometries modeled on $E^2$. For $T^2$, we can see this by taking $\Gamma$ to be the group of isometries of $E^2$ generated by two translations in independent directions. We will then have $T^2 \simeq E^2/\Gamma$. 

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There are infinitely many surfaces of Euler characteristic less than zero, and it turns out that all are homeomorphic to $H^2/\Gamma$ for some discrete group of isometries of $H^2$. This is most easily seen by first putting a hyperbolic structure on $3P^2$, the surface of Euler characteristic $-1$, and then observing that any surface with negative Euler characteristic covers $3P^2$. For a concrete example, though, we look to $2T^2$, the surface of genus 2. Consider the standard presentation of $2T^2$ to be a regular octagon in $H^2$. It turns out that the four maps which identify pairs of edges of this presentation are isometries of $H^2$. Letting $\Gamma$ be the group of isometries of $H^2$ generated by these four maps, we have that $2T^2 \simeq H^2/\Gamma$.

2 The Eight 3-Dimensional Geometries

From the two-dimensional results discussed above, it is natural to wonder whether an analogous result to Uniformization Theorem holds for 3-manifolds and the geometries $S^3$, $E^3$, and $H^3$. It turns out not to be the case. For instance, $S^2 \times S^1$ has universal covering $S^2 \times \mathbb{R}$, which is not homeomorphic to any of $S^3$, $E^3$, $H^3$. A 3-dimensional result will require more work, and in particular more model geometries. We describe them below.

2.1 $E^3$

$E^3$ is just 3-dimensional Euclidean space, defined analogously to $E^2$. Its isometry group is the set of maps $E^3 \rightarrow E^3$, $x \mapsto ax + b$, where $a \in O(3)$ and $b \in E^3$. That is, every isometry of $E^3$ is a rotation or reflection followed by a translation.

It turns out that there are ten compact 3-manifolds which are modeled on $E^3$ (and are therefore flat). One is the 3-torus, which is homeomorphic to $E^3/\Gamma$ for the discrete group $\Gamma$ of isometries generated by three translations in independent directions. The other nine are finitely covered by the 3-torus.

2.2 $S^3$

$S^3$ is of course just the 3-sphere with the metric induced by the Euclidean metric of $E^4$. Like its 2-dimensional counterpart, $S^3$ has constant positive curvature 1. The full group of isometries is $O(4)$, the group of rotations in $E^4$.

For an example of a manifold which possesses a geometric structure modeled on $S^3$, consider the following construction. Consider $S^3$ to be the unit sphere in $\mathbb{C}^4$, i.e., $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$. Fix relatively prime $p, q \in \mathbb{Z}$. Then it is not hard to see that the map $f_{p,q} : S^3 \rightarrow S^3$, $(z_1, z_2) \mapsto (e^{2\pi i \frac{p}{q}} z_1, e^{2\pi i \frac{q}{p}} z_2)$, is an isometry of $S^3$. The lens space $L_{p,q}$ is defined to be $L_{p,q} = S^3/\langle f_{p,q} \rangle$, so that $L_{p,q}$ has a geometry modeled on $S^3$. In particular, $L_{p,q}$ is a manifold (since $\langle f_{p,q} \rangle$ acts freely and properly discontinuously) with constant positive curvature.

2.3 $H^3$

$H^3$ is the 3-dimensional model of hyperbolic space, defined (analogously to $H^2$) to be the upper-half space with the metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$ 

$H^3$, like $H^2$, has constant negative curvature $-1$, and geodesics are vertical lines and arcs of circles which intersect the $xy$-plane orthogonally.

Typical examples of hyperbolic 3-manifolds (i.e., spaces with geometries modeled on $H^3$) are knot complements, such as the famous figure-eight knot complement.

2.4 $S^2 \times \mathbb{R}$

Endowed with the product metric, this is one of the least interesting of the spaces on this list. Its group of isometries can be identified with $\text{Isom} S^2 \times \text{Isom} \mathbb{R}$.

There are just seven 3-manifolds with geometric structure modeled on $S^2 \times \mathbb{R}$. Unsurprisingly, one is $P^2 \times \mathbb{R}$, which is the quotient of $S^2 \times \mathbb{R}$ by the group of order 2 generated by the pair $(\alpha, \beta)$, where $\alpha$ is the
antipodal map of $S^2$ and $\beta$ is the identity on $\mathbb{R}$. More interestingly, it turns out that $P^3 \# P^3$ has a geometric structure modeled on $S^2 \times \mathbb{R}$. Taking $\Gamma = \langle (\alpha, \beta_1), (\alpha, \beta_2) \rangle \subseteq \text{Isom } S^2 \times \mathbb{R}$, where $\alpha$ is the antipodal map of $S^2$ and $\beta_1, \beta_2$ are distinct reflections of $\mathbb{R}$, we have $P^3 \# P^3 \simeq (S^2 \times \mathbb{R})/G$. In fact, $P^3 \# P^3$ is the only closed 3-manifold which admits a geometric structure and can be written as a nontrivial connected sum. See Scott [1] for details.

2.5 $H^2 \times \mathbb{R}$

$H^2 \times \mathbb{R}$ is also endowed with the product metric, and its full isometry group can be identified with $\text{Isom } H^2 \times \text{Isom } \mathbb{R}$.

From our discussion of $H^2$, it is clear that there are infinitely many 3-manifolds with geometric structure modeled on $H^2 \times \mathbb{R}$. In particular, any product $M \times \mathbb{R}, M \times S^1$ for $M$ a hyperbolic 2-manifold has a geometric structure modeled on $H^2 \times \mathbb{R}$.

2.6 $\widetilde{SL}_2 \mathbb{R}$

We must now also consider some more exotic spaces. First among them is $\widetilde{SL}_2 \mathbb{R}$, the universal cover of the space of $2 \times 2$ real matrices with determinant 1. $\widetilde{SL}_2 \mathbb{R}$ itself has a group structure that makes it a 3-dimensional Lie group. The description of its metric is somewhat complicated compared to the other spaces on this list, so we simply say that, given its Lie group structure, $\widetilde{SL}_2 \mathbb{R}$ admits a left-invariant metric.

2.7 Nil

Nil, also known as the Heisenberg group, is defined by

$$\text{Nil} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. $$

It is a three-dimensional Lie group with left-invariant metric (in the above coordinates) $ds^2 = dx^2 + dy^2 + (dz - xdy)^2$.

2.8 Sol

Finally, there is Sol, the solvable group. As a group, it is defined to be $\mathbb{R} \times \mathbb{R}^2$, with $t \in \mathbb{R}$ acting on $\mathbb{R}^2$ by $(x, y) \mapsto (e^tx, e^{-t}y)$. Sol is a 3-dimensional Lie group with left-invariant metric Sol is $ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dz^2$.

2.9 A Classification of Thurston

Now, we solidify our notion of a geometric structure so that we may state a landmark result of Thurston.

**Definition.** A geometry is a pair $(X, G)$, where $X$ is a Riemannian manifold and $G$ is a group of isometries of $X$ acting transitively and with compact point stabilizers. A geometry $(X, G)$ is called maximal if $G$ is maximal among subgroups of Isom $X$ which have compact point stabilizers. Two geometries $(X, G)$ and $(X', G')$ are equivalent if there is a diffeomorphism $X \to X'$ which throws the action of $G$ into the action of $G'$.

With this definition, we have the following:

**Theorem (Thurston).** Any maximal, simply-connected, 3-dimensional geometry which admits a compact quotient is equivalent to one and only one of the geometries $(X, \text{Isom } X)$, where $X$ is one of $E^3$, $H^3$, $S^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL}_2 \mathbb{R}$, Nil, or Sol.
3 The Geometrization Theorem

One statement of the theorem is as follows:

**Theorem (Geometrization Theorem).** Any compact, orientable 3-manifold \( M \) can be cut by disjoint embedded 2-spheres and tori into pieces which, after gluing 3-balls to all boundary spheres, admit geometric structures.

It turns out that, along with a few other results, the Geometrization Theorem implies the legendary Poincaré Conjecture:

**Corollary (Poincaré Conjecture).** Any simply-connected closed smooth 3-manifold is diffeomorphic to \( S^3 \).

4 Proving the Geometrization Theorem

In this section, we try to give the very broad idea employed in the proof of the Geometrization Theorem. The discussion will be rather heuristic.

4.1 The Ricci Flow

The **Ricci flow** is a means of evolving the metric \( g \) of a Riemannian manifold over “time” through the following differential equation:

\[
\frac{\partial}{\partial t} g = -2Rc, \quad g(0) = g_0,
\]

where \( g_0 \) is the initial metric on the manifold and \( Rc \) is the Ricci tensor of the manifold (which also evolves in time due to its dependence on the metric). This equation can result in curvature singularities in finite time. For instance, under the Ricci flow, the 3-sphere will shrink to a point in finite time. To fix this, one can do a rescaling to obtain the **normalized Ricci flow**,

\[
\frac{\partial}{\partial t} g = -2Rc + \frac{2}{n} \int_M R \, d\mu, \quad g(0) = g_0,
\]

where \( n \) is the dimension of \( M \) (we are of course interested in \( n = 3 \)). Hamilton first introduced the Ricci flow in his landmark paper [2] with geometrization in mind. He used it to prove the following theorem:

**Theorem (Hamilton).** Let \( X \) be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then \( X \) also admits a metric of constant positive curvature.

In particular, under the normalized Ricci flow, 3-manifolds of strictly positive curvature evolve to the 3-sphere, one of the eight Thurston geometries.

For a general 3-manifold \( M \), the idea of the Ricci flow program for geometrization is that the flow will evolve pieces of \( M \) of “like” curvature to the model geometries on Thurston’s list, and that the pieces that the flow acts can be put back together along embedded 2-spheres and tori, just as in the statement of the theorem. However, curvature singularities can occur, and they are not as easily dealt with as in the case of the sphere. The curvature blows up in finite time everywhere on the sphere, a problem that was easily corrected by a rescaling of time. However, on more general manifolds, the Ricci flow may give rise to curvature singularities in finite time on some parts of the manifold but not others. A simple rescaling will then be of little use. Perelman’s achievement was to understand the types of possible singularities well enough to work around them in what has been dubbed “Ricci flow with surgery”.

References


