Differential Forms, Integration on Manifolds, and Stokes’ Theorem

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Introduction: Theorems of Classical Calculus

The Fundamental Theorem of Calculus

\[ \int_{a}^{b} f(x) \, dx = f(b) - f(a) \]

The Fundamental Theorem for Line Integrals

\[ \int_{C} \nabla f \cdot dr = f(r(b)) - f(r(a)) \]
Introduction: Theorems of Classical Calculus

**Green’s Theorem**

\[ \iint_{A} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_{\partial A} (P \, dx + Q \, dy) \]

**Stokes’ Theorem**

\[ \iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \]
The Divergence Theorem (Gauss’ Theorem)

\[ \iiint_V (\nabla \cdot \mathbf{F}) \, dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} \]
A linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *derivation at* $p \in M$ if, for all $f, g \in C^\infty(M)$,

$$X(fg) = f(p)X(g) + g(p)X(f)$$

The set of all derivations of $C^\infty(M)$ at $p$ is a vector space called the *tangent space* to $M$ at $p$, denoted by $T_pM$. 
Covectors

**Definition**

Let $p \in M$. The *cotangent space* $T_p^*M$ of $M$ at $p$, is defined to be the dual of the tangent space at $p$,

$$T_p^*M = (T_pM)^*$$

An element $\omega \in T_p^*M$, called a *covector*, is a linear map $\omega : T_pM \rightarrow \mathbb{R}$. 

The Differential of a Function

Definition

Let $f : M \to \mathbb{R}$ be smooth. We define a covector field $df$, called the *differential* of $f$, by

$$df_p(X_p) = X_p f$$

for all $X_p \in T_p M$.

In coordinates, we can write

$$df = \frac{\partial f}{\partial x^i} dx^i$$
Properties of the Differential

The differential satisfies many the properties we would expect it to:

**Lemma**

Let $f, g : M \to \mathbb{R}$ be smooth. Then:

- For any constants $a, b$, $d(af + bg) = adf + b dg$.
- $d(fg) = f dg + g df$.
- $d(f/g) = (g df - f dg)/g^2$ on the set where $g \neq 0$.
- If $J \subseteq \mathbb{R}$ is an interval containing $f(M)$ and $h : J \to \mathbb{R}$ is smooth, then $d(h \circ f) = (h' \circ f) df$.
- If $f$ is constant, then $df = 0$. 
**Tensors**

**Definition**

Let $V$ be vector space. A *covariant $k$-tensor* on $V$ is a real-valued multilinear function of $k$ elements of $V$:

$$T : V \times \cdots \times V \rightarrow \mathbb{R}$$

**Examples:**

- The metric $g$ of a Riemannian manifold is a covariant 2-tensor.
- In classical electrodynamics, the *electromagnetic field tensor* $F$ is given (in coordinates) by

$$F_{\mu\nu} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix}$$
Tensor Product

**Definition**

Let $V$ be a finite-dimensional real vector space and let $S \in T^k(V)$, $T \in T^l(V)$. Define a map

$$S \otimes T : V \times \cdots \times V \rightarrow \mathbb{R}$$

by

$$S \otimes T(X_1, \ldots, X_{k+l}) = S(X_1, \ldots, X_k) T(X_{k+1}, \ldots, X_{k+l})$$
A covariant $k$-tensor $T$ on a finite-dimensional vector space $V$ is said to be *alternating* if

$$T(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k) = -T(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k)$$
Given a covariant $k$-tensor $T$, we define the alternating projection of $T$ to be the covariant $k$-tensor

$$\text{Alt} T = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn} \, \sigma)(^\sigma T)$$
Example - Alternating Projection

1. If $T$ is a 1-tensor, then

   \[ \text{Alt} \ T = T \]

2. If $T$ is a 2-tensor, then

   \[ \text{Alt} \ T(X, Y) = \frac{1}{2} (T(X, Y) - T(Y, X)) \]

3. If $T$ is a 3-tensor, then

   \[ (\text{Alt} \ T)_{ijk} = \frac{1}{6} (T_{ijk} + T_{jki} + T_{kij} - T_{jik} - T_{ikj} - T_{kji}) \]
A differential $k$-form is a continuous tensor field whose value at each point is an alternating tensor.
The Wedge Product

We want a way to produce new differential forms from old ones:

**Definition**

Given a \( k \)-form \( \omega \) and an \( l \)-form \( \eta \), we define the *wedge product* or *exterior product* of \( \omega \) and \( \eta \) to be the \((k + l)\)-form

\[
\omega \wedge \eta = \frac{(k + l)!}{k!l!} \text{Alt}(\omega \otimes \eta)
\]

**Some Properties of the Wedge Product**

- **Bilinearity**
- **Associativity**
- **Anticommutativity:**

\[
\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega
\]
The Exterior Derivative

Theorem

For every smooth manifold $M$, there are unique linear maps $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ defined for each integer $k \geq 0$ and satisfying the following three conditions:

- If $f$ is a smooth real-valued function (a 0-form), then $df$ is the differential of $f$.
- If $\omega \in \mathcal{A}^k$ and $\eta \in \mathcal{A}^l$, then
  
  \[ d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \]

- $d^2 = d \circ d = 0$. 
In coordinates,

$$d \left( \sum'_j \omega_j dx^j \right) = \sum'_j d\omega_j \wedge dx^j$$

where $d\omega_j$ is just the differential of the function $\omega_j$. 
Orientations of Vector Spaces

**Definition**

Any two bases \((E_1, \ldots, E_n)\) and \((\tilde{E}_1, \ldots, \tilde{E}_n)\) of a finite-dimensional vector space \(V\) are related by a transition matrix \(B = (B_i^j)\),

\[ E_i = B_i^j \tilde{E}_j \]

We say that \((E_1, \ldots, E_n)\) and \((\tilde{E}_1, \ldots, \tilde{E}_n)\) are *consistently ordered* if \(\det(B) > 0\).

"Consistently ordered" defines an equivalence relation on the set of all (ordered) bases of \(V\). There are exactly two equivalence classes, which we refer to as orientations of \(V\). A vector space along with a choice of orientation is called an oriented vector space. The *standard orientation* of \(\mathbb{R}^n\) is \([e_1, \ldots, e_n]\).
Orientations of Arbitrary Manifolds

Definition

A pointwise orientation of a manifold $M$ is just a choice of orientation of each tangent space. An orientation of $M$ is a continuous pointwise orientation. $M$ is said to be orientable if there exists an orientation for it.
Manifolds with Boundary

Informally, an $n$-dimensional manifold with boundary is a space which is "like" $\mathbb{R}^n$ except at certain boundary points. Formally,

**Definition**

An $n$-dimensional topological manifold with boundary is a second-countable Hausdorff space $M$ in which every point has a neighborhood homeomorphic to an open subset $U$ of $H^n$, where $H^n$ is the upper half-space,

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n \geq 0\}.$$

Examples:

- The unit interval, $[0, 1]$.
- Closed balls $\bar{B}_r(x)$ in $\mathbb{R}^n$. 

First, a little terminology:

**Definition**

The *support* of an $n$-form is $\omega$ is the closure of the set $\{p \in M | \omega(p) \neq 0\}$. $\omega$ is said to be *compactly supported* if its support is a compact set.

**Definition**

Let $\omega = f dx^1 \cdots dx^n$ be an $n$-form on $\mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ be compact. We define

$$\int_D \omega = \int_D f \ dx^1 \cdots dx^n$$
Definition

Let $M$ be a smooth, oriented $n$-manifold, and let $\omega$ be an $n$-form on $M$. If $\omega$ is compactly supported in the domain of a single oriented coordinate chart $(U, \phi)$, we define the integral of $\omega$ over $M$ to be

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega$$

If we require more than a single chart to cover the support of $\omega$, then, informally speaking, $\int_M \omega$ is the sum of the integrals over each chart, minus overlap.
Stokes’ Theorem

Theorem

Let $M$ be a smooth, oriented $n$-dimensional manifold with boundary, and let $\omega$ be a compactly supported smooth $(n-1)$-form on $M$. Then

$$\int_M d\omega = \int_{\partial M} \omega$$
The Fundamental Theorem for Line Integrals

**Theorem**

Let \( C : [a, b] \rightarrow \mathbb{R}^n \) be a smooth curve in \( \mathbb{R}^n \) such that \( M = C([a, b]) \) is a 1-dimensional submanifold with boundary of \( \mathbb{R}^n \). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be smooth. Then

\[
\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))
\]