When a company manufactures a product, their "cost analysis team" creates a cost function \( C(x) \), which is intended to measure the cost of producing \( x \) units. Simple example:

\[
C(x) = 100 + 10x + x^2
\]

1. \( C(0) = 100 \), $100 can be interpreted as startup costs.

2. \( C(10) = 100 + 100 + 100 = 300 \)

3. \( C(100) = 100 + 1000 + 10000 = 11,100 \)

The average cost of making \( x \) units is

\[
a(x) = \frac{C(x)}{x}
\]

So

\[
a(10) = \frac{300}{10} = $30 \text{ per unit.}
\]

\[
a(100) = \frac{11,100}{100} = $111 \text{ per unit.}
\]
The next unit cost is the cost of making one more unit after making the first $x$ units: $N(x) = C(x+1) - C(x)$. So the cost of the 15th unit is $N(15) = C(16) - C(15)$:

$$N(15) = 516 - 475$$

$$N(15) = 41$$

There is another approach to the next unit. Recall that $C'(x) = \lim\limits_{h \to 0} \frac{C(x+h) - C(x)}{h} \approx \frac{C(x+h) - C(x)}{h}$ (if $h$ is small, relative to $x$).

Since most manufacturing processes create thousands of units, then $h=1$ would be small relative to $x$, so

$$C'(x) \approx \frac{C(x+1) - C(x)}{1} = N(x).$$

Thus $C'(x)$ is a good approximation to $N(x)$:

$$C(x) = 100 + 10x + x^2 \implies C'(x) = 10 + 2x$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$C'(x)$</th>
<th>$N(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
<td>31</td>
</tr>
<tr>
<td>15</td>
<td>40</td>
<td>41</td>
</tr>
<tr>
<td>100</td>
<td>210</td>
<td>211</td>
</tr>
<tr>
<td>1000</td>
<td>2010</td>
<td>2011</td>
</tr>
</tbody>
</table>

$C'(x)$ is called the Marginal Cost.
Another example: \[ C(x) = 2600 + 2x + (0.001)x^2 \]
\[ C'(x) = \frac{C(x)}{x} = \frac{2600}{x} + 2 + (0.001)x \]
\[ C'(x) = 2 + (0.001)x \]

<table>
<thead>
<tr>
<th>X</th>
<th>C(X)</th>
<th>A(X)</th>
<th>C'(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>5.600</td>
<td>5.60</td>
<td>4.00</td>
</tr>
<tr>
<td>1500</td>
<td>7.850</td>
<td>5.23</td>
<td>5.06</td>
</tr>
<tr>
<td>2000</td>
<td>10.800</td>
<td>5.32</td>
<td>6.00</td>
</tr>
<tr>
<td>2500</td>
<td>13.850</td>
<td>5.54</td>
<td>7.00</td>
</tr>
<tr>
<td>3000</td>
<td>17.600</td>
<td>5.87</td>
<td>8.00</td>
</tr>
</tbody>
</table>

It appears that \( a(x) \) has a minimum value, which usually would be found at a critical value of \( a(x) \). That is, we set \( a'(x) = 0 \) and solve. In this case, there is a trick.

\[ a(x) = \frac{C(x)}{x} \]
\[ a'(x) = \frac{C'(x) \cdot x - C(x) \cdot 1}{x^2} = 0 \]
\[ C'(x) \cdot x - C(x) = 0 \]
\[ C'(x) \cdot x = C(x) \]
\[ C'(x) = \frac{C(x)}{x} \]

Thus, the minimal value of \( a(x) \) occurs for the value of \( x \) that makes marginal cost = average cost.
So we will compute the production level $x$ that minimizes average cost.

Set $C'(x) = 0$,

\[ x^2 + (0.002)x = \frac{21000}{x} + x + (0.001)x \]

\[ 0.001x = \frac{21000}{x} \]

\[ x^2 = \frac{21000}{0.001} \]

\[ x^2 = 21,000,000 \]

\[ x = \sqrt{21,000,000} \]

\[ x \approx 1612 \] — production level with least over cost.

Average cost: $a(1612) = \$5.22 \text{ /unit}$.
Marketing

Price function: $p(x) = \text{maximum price that can be charged per unit if you want to sell } x \text{ units.}$

Revenue function: $R(x) = x \cdot p(x)$ is the revenue obtained by selling $x$ unit at $p(x)$ per unit.

Profit function: $P(x) = R(x) - C(x)$

OBJECTIVE: MAXIMIZE PROFIT

The maximum of $P(x)$ will usually occur at a critical value of $p(x).$ Again a trick:

$P(x) = R(x) - C(x)\\ P'(x) = R'(x) - C'(x) = 0

\boxed{R'(x) = C'(x)}$

Thus the profit is largest for the production level $x$ that makes marginal revenue = marginal cost
Example: Determine the production level that will maximize the profit for a company with cost and price functions

\[ C(x) = 94 + (1.26)x - (0.01)x^2 + (0.00007)x^3 \]
\[ p(x) = 3.5 - (0.01)x \]

\[ R(x) = x \cdot p(x) = (3.5)x - (0.01)x^2 \]
\[ R'(x) = 3.5 - (0.02)x \quad \text{— marginal revenue} \]
\[ C'(x) = 1.26 - (0.02)x + (0.00021)x^2 \quad \text{— marginal cost} \]

Set \( R'(x) = C'(x) \):

\[ 3.5 - (0.02)x = 1.26 - (0.02)x + (0.00021)x^2 \]

\[ x^2 = \frac{2.24}{0.00021} = 10666.66 \]

\[ x = 103 \quad \text{— maximum profit}. \]

\[ P(x) = R(x) - C(x) \]
\[ P(103) = R(103) - C(103) \]
\[ P(103) = 254.41 - 184.18 \]
\[ P(103) = 70.23 \]

NOTE: Price Functions often take the form:

\[ p(x) = a - b \cdot x \]
Revenue Example: A surplus store has purchased a large number (thousands) of DVD players from a bankrupt store. They are selling 200 per week at a price of $350 each. This generates a weekly revenue of $70,000. A market survey shows that a $10 decrease in price will lead to 20 more sales per week. How much should they drop the price to maximize weekly revenue.

The price function is \( p(x) = a - bx \), with \( p(200) = 350 \) and \( p(220) = 340 \)

\[
\begin{align*}
    a - 200b &= 350 \\
    a - 220b &= 340
\end{align*}
\]

\[
\Rightarrow 20b = 10 \quad \Rightarrow b = \frac{1}{2}
\]

\[
a = 450
\]

\[
p(x) = 450 - \frac{1}{2}x
\]

\[
R(x) = 450x - \frac{1}{2}x^2
\]

Maximize \( R(x) \): \( R' = 450 - x = 0 \)

\[
x = 450 \quad \text{maximizes revenue}
\]

\[
p(450) = 225 \quad \text{price to charge}
\]

\[
R(450) = 450 \cdot 225 = 101,250
\]
PROBLEM 1

[5] (a) Find the cost, average cost and marginal for a production run of \( x = 1000 \) units.
(b) Find the production level that minimizes average cost
(c) Find the minimal average cost.

\[ \text{(Econ. 2 #5)} \]

\[ C(x) = 40000 + 300x + x^2 \]  
- cost [This is given in the problem]

\[ C'(x) = 300 + 2x \]  
- marginal cost

a)  
\[ C(1000) = 40000 + 300(1000) + (1000) \]
\[ = 40000 + 300000 + 1000000 = 1,340,000 \]

\[ a(x) = \frac{C(x)}{x} = \frac{40000}{x} + 300 + x \]

\[ a(1000) = \frac{40000}{1000} + 300 + 1000 = \$340 \]

\[ C'(1000) = 300 + 2(1000) = \$2300 \]

b)  
\[ a(x) \text{ is minimal when } C'(x) = a(x) : \]
\[ 300 + 2x = \frac{40000}{x} + 300 + x \]
\[ x = \frac{40000}{x} \]
\[ x^2 = 40000 \]
\[ x = \$200 \]

C)  
\[ a(200) = \frac{40000}{200} + 300 + 200 = 200 + 300 + 200 \]
\[ a(200) = \$700 \text{ (unit)} \]
PROBLEM 2

#14 Find the production level that will maximize profit.

\[ C(x) = 16000 + 500x - 1.6x^2 + 0.004x^3 \quad \text{cost \ [Given]} \]

\[ p(x) = 1700 - 7x \quad \text{demand \ (price) \ function \ [Given]} \]

\[ R(x) = xp(x) = 1700x - 7x^2 \quad \text{revenue} \]

\[ C'(x) = 500 - 3.2x + 0.012x^2 \quad \text{Marginal cost} \]

\[ R'(x) = 1700 - 14x \quad \text{Marginal revenue} \]

Profit is maximized when \( R'(x) = C'(x) \):

\[ 500 - 3.2x + 0.012x^2 = 1700 - 14x \]

\[ 0.012x^2 + 10.8x - 1200 = 0 \]

\[ 12x^2 + 1080x - 120000 = 0 \]

\[ 900x - 100000 = 0 \]

\[ (x-100)(x+1000) = 0 \]

\[ x = 100 \quad \text{(can’t have neg. x)} \]