Extreme Values

If the input, $x$, of a smooth function, $f(x)$, moves through a finite closed interval, there will be at least one absolute maximum value and at least one absolute minimum value taken by $f(x)$ on the interval.

Examples:

1. 

2. 

3. 

4. 

The min and max points are called the extreme points of the function in the specified interval.

Extreme points can occur at only two places.

1. End points
2. Critical points (where $f'(x) = 0$). These are the reversal points.
Find the extreme points of \( f(x) = 2x^3 - 3x^2 - 12x \) on \([-2, 4]\).

Find critical points: \( f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) \)

\[
\begin{align*}
    f'(x) &= 0 \\
    x^2 - x - 2 &= 0 \\
    (x-2)(x+1) &= 0 \\
    x &= -1, 2
\end{align*}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
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<tbody>
<tr>
<td>end</td>
<td>-2</td>
</tr>
<tr>
<td>crit</td>
<td>-1</td>
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<tr>
<td>crit</td>
<td>2</td>
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<td>end</td>
<td>4</td>
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Find the extreme points of \( g(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) on \([-2, 3]\).

Critical: \( g'(x) = 12x^3 - 12x^2 - 24x = 12(x^3 - x^2 - 2x) \)

\[
\begin{align*}
    g'(x) &= 0 \\
    x(x^3 - x^2 - 2x) &= 0 \\
    (x+1)(x^2 - x - 2) &= 0 \\
    x &= -1, 0, 2
\end{align*}
\]

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<td>0</td>
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<td>end</td>
<td>3</td>
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A square of cardboard is 3 ft by 3 ft. A piece will be cut from each corner (a small square) and the remaining material will be bent up to form a square-bottom open-top box. How big should the corner cut be in order to obtain a maximum volume?

\[ V = l^2 x. \]

Constraint:
\[
\begin{align*}
2x + l &= 3 \\
l &= 3 - 2x \\
l^2 &= 9 - 12x + 4x^2
\end{align*}
\]

\[ V(x) = l^2 x = (9 - 12x + 4x^2) x \]

\[ V(x) = 4x^3 - 12x^2 + 9x \]

Clearly: \( 0 \leq x \leq 3/2 \)

So we want to find extreme values of \( V(x) \) on \([0, 3/2] \).

\[
\begin{array}{c|c|c}
\text{X} & \text{V} \\
\hline
\text{End} & 0 & 0 \\
\text{Crit} & 1/2 & 2 \\
\end{array}
\]

Make box 2x2x1/2.

Max Vol = 2 cubic ft.
Optimization - minimal value.

Two towns (T1 and T2) lie south of a large lake. See the diagram below. The towns want to build a pumping station, P, on the south shore to send lake water to each town. Where should P be placed in order to minimize the total pipe required to reach from P to the towns?

Find critical values of \( L(x) \):

\[
L'(x) = \frac{2x}{\sqrt{x^2 + 1}} + \frac{2x - 8}{\sqrt{2(x^2 - 8x + 32)}}
\]

\( L'(x) = 0 \)

Equation reduces to (see next page)

\[
15x^2 + 8x - 16 = 0
\]

\((5x-4)(3x+4) = 0\)

\( x = -4/3, 4/5 \)

But \(-4/3\) is not in \([0,4]\)

\[
\begin{array}{c|c}
 x & L(x) \\
 0 & 6.657 \\
 4/5 & 6.403 \\
 4 & 8.123 \\
\end{array}
\]

\( \leftarrow \text{min} \)

Total pipe:

\[
L(x) = L_1 + L_2
\]

\[
L_1 = \sqrt{x^2 + 1} \quad L_2 = \sqrt{(4-x)^2 + 4^2}
\]

\[
L(x) = \sqrt{x^2 + 1} + \sqrt{x^2 - 8x + 32}
\]

where \( x \) is restricted to \([0,4]\).
Details:

\[ L(x) = \sqrt{x^2+1} + \sqrt{x^2-8x+32} \]

\[ L'(x) = \frac{2x}{2\sqrt{x^2+1}} + \frac{2x-8}{2\sqrt{x^2-8x+32}} = \frac{x}{\sqrt{x^2+1}} - \frac{(4-x)}{\sqrt{x^2-8x+32}} \]

\[ L'(x) = 0 \]

\[ \frac{x}{\sqrt{x^2+1}} = \frac{4-x}{\sqrt{x^2-8x+32}} \]

\[ x\sqrt{x^2-8x+32} = (4-x)\sqrt{x^2+1} \]

\[ x^2(x^2-8x+32) = (4-x)^2(x^2+1) \]

\[ x^4 - 8x^3 + 32x^2 = x^4 - 8x^3 + 17x^2 - 8x + 16 \]

\[ 32x^2 = 17x^2 - 8x + 16 \]

\[ 15x^2 + 8x - 16 = 0 \]

\[ (5x-4)(3x+4) = 0 \]

\[ x = \frac{4}{5} \]

\[ x = -\frac{4}{3} \]
Optimization - minimal value

A man lives in a house (H) on the south side of a lake. He likes to paddle his canoe across to a cafe (C) for lunch. He paddles at a rate \( v_1 = 4 \, \text{ft/sec} \) and walks at a rate of \( v_2 = 6 \, \text{ft/sec} \). How should he plan his trip to reach the cafe in minimal time?

![Diagram of the scenario]

Trip: row to point P, distance = \( d_1 \), time = \( T_1 \)  
walk from P to C, distance = \( d_2 \), time = \( T_2 \)

\[ T_1 = \frac{d_1}{v_1} = \frac{\sqrt{x^2 + 600^2}}{4} \quad T_2 = \frac{d_2}{v_2} = \frac{2000 - x}{6} \]

Total time \( T(x) = T_1 + T_2 = \frac{\sqrt{x^2 + 600^2}}{4} + \frac{2000 - x}{6}, \quad 0 \leq x \leq 2000 \).  

\[ T' = \frac{x}{4\sqrt{x^2 + 600^2}} - \frac{1}{6} = 0 \]

\[ 3x = 2 \sqrt{x^2 + 600^2} \]

\[ 9x^2 = 4x^2 + 4(600^2) \]

\[ 5x^2 = 4(600^2) \]

\[ x = \frac{1200}{\sqrt{5}} \approx 536.656 \, \text{ft} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( T )</th>
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<tbody>
<tr>
<td>0</td>
<td>3.05 min</td>
<td></td>
</tr>
<tr>
<td>536.656</td>
<td>7.42 min</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>8.70 min</td>
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