Last time we asked the question of what happens if a subset $S$ of a vector space $V$ isn’t a subspace? What can we add to $S$ in order to make it a subspace? To answer this question we had to introduce the notion of a linear combination of the subset $S$. We went on to show that the smallest subspace of a vector space containing the subset $S$ is the set of all linear combinations of $S$, which we call the span of $S$ and denote $\text{span}(S)$.

Today we’re going to look at linear combinations of a set $S$ from the opposite perspective. Specifically, given a vector $v$ and a set $S$, how can we determine if $v$ is a linear combination of the set $S$? To answer this question we will need to solve systems of linear equations.

We will then consider the problem of removing redundant vectors from a subset $S$ of a vector space. By redundant, we mean that these vectors can be removed and the resulting set will have the same span as the original set. We will then introduce the notion of linear dependence and independence of vectors.

1 Linear dependence and independence

Let $V$ be a vector space and $S$ a subset of $V$. Since every vector in $\text{span}(S)$ is a linear combination of vectors in $S$, we can think of the vectors of $S$ as the “building blocks” of the subspace $\text{span}(S)$. For example, consider the subset

$$ S = \{(1,0,0), (0,1,0), (2,3,0)\} $$

of $\mathbb{R}^3$. The span of this set $\text{span}(S)$ is the entire $xy$-plane pictured below:
In other words, if you pick any point on the $xy$-plane above, that point can be written as a linear combination of the vectors $(1,0,0)$, $(0,1,0)$, and $(2,3,0)$.

Now it may be the case that some of these building blocks are actually unnecessary, that is, if we were to remove them from $S$ the resulting set would have the same span. For example, we could remove the vector $(2,3,0)$ from the set above, and we still have $\text{span}([1,0,0],[0,1,0])$ is all of the $xy$-plane. So the vector $(2,3,0)$ wasn’t a necessary “building block”. It’s natural to want to remove such vectors from our set and reduce our set $S$ down to just those vectors that are to necessary to generate the subspace $\text{span}(S)$.

Let’s think about what this means. Suppose $v \in S$ and let $S - \{v\}$ denote the set $S$ with the vector $v$ removed. Since $v$ is clearly in $\text{span}(S)$, if $\text{span}(S) = \text{span}(S - \{v\})$ it would mean that $v$ can be written as a linear combination of the set $S - \{v\}$. That is, there are a finite number of vectors $v_1, \ldots, v_n \in S - \{v\}$ and scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$v = a_1v_1 + \cdots + a_nv_n. \quad (1)$$

In other words a vector $v \in S$ is redundant or unnecessary if it is a linear combination of the other vectors in $S$. So, to determine if a vector $v \in S$ is redundant, we could just determine if the linear system resulting from $(1)$ has a solution. However, there is a problem with this approach - in order to remove redundant vectors you may have to solve many linear systems if your set $S$ is large. This could
be prohibitively time consuming. Luckily there is a more efficient solution.

The existence of the linear combination (1) is equivalent to the existence of the linear combination

\[-v + a_1v_1 + \cdots + a_nv_n = 0.\]  

Furthermore, note that the above linear combination would imply that each \(v_i\) with \(a_i \neq 0\) is redundant. Therefore, by solving just one linear system (2) we can gain as much information as we would solving (1) for each vector \(v, v_1, \ldots, v_n\). To make this more concrete let’s look at an example.

**Example 1.** Consider the subset of vectors

\[
{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)}
\]

from the vector space \(\mathbb{R}^4\).

Suppose we want to determine whether \((-1, 0, 1, 0)\) is a linear combination of the remaining vectors

\[
{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4)}.\]

To determine this requires us to solve the following linear system:

\[
\begin{align*}
3a_1 + 2a_2 + a_3 &= -1 \\
-4a_1 + 2a_2 - 3a_3 &= 0 \\
-4a_1 - 4a_2 + 2a_3 &= 1 \\
2a_1 - 4a_3 &= 0.
\end{align*}
\]

Subtracting suitable multiplies of the first row from the second, third, and fourth, we get

\[
\begin{align*}
3a_1 + 2a_2 + a_3 &= -1 \\
-4a_2 - 6a_3 &= 3 \\
8a_2 + 6a_3 &= -3 \\
-4a_2 - 6a_3 &= 2.
\end{align*}
\]

And we can stop here. Note that the left hand side of the second and fourth equations are the same, but the right hand sides are not. If we were to subtract that two, we would get \(0 = -1\) or \(0 = 1\), depending on which equation you subtract
from which. Regardless, they are both nonsense and tell us that there are no \( a_1, a_2, \) and \( a_3 \) satisfying the above equations. That is, \((-1,0,1,0)\) is not a linear combination of the remaining vectors \(\{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4)\}\).

Now, in order to determine if any of the other vectors are a linear combination of the complimentary vectors we would have to repeat this process again. Ugh.

Let’s try the other approach. In this case we solve the linear system:

\[
\begin{align*}
a_1 + 2a_2 + a_3 - a_4 &= 0 \\
3a_1 + 2a_2 - 3a_3 &= 0 \\
-4a_1 - 4a_2 + 2a_3 + a_4 &= 0 \\
2a_1 - 4a_3 &= 0
\end{align*}
\]

Solving this we would find that \( a_1 = 4, a_2 = -3, a_3 = 2, \) and \( a_4 = 0 \) is a solution. That is,

\[
4(1,3,-4,2) + -3(2,2,-4,0) + 2(1,-3,2,-4) + 0(-1,0,1,0) = \vec{0}.
\]

Not only does \( a_4 = 0 \) tell us that \((-1,0,1,0)\) is not a linear combination of the other vectors \(\{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4)\}\).

It tells us that each vector in the set \(\{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4)\}\) can be written as a linear combination of the others.

To summarize, if the zero vector \( \vec{0} \) can be expressed as a nontrivial linear combination of a set of vectors from \( S \), then there are redundant vectors in the set \( S \). Here, nontrivial means that at least one of the scalars in the linear combination must be nonzero. This is such an important concept we give it’s own name.

**Definition 1.1.** A subset \( S \) of a vector space \( V \) is said to be linearly dependent if there exist a finite number of vectors \( v_1, \ldots, v_n \in S \) and scalars \( a_1, \ldots, a_n \in \mathbb{R} \), with at least one \( a_i \neq 0 \), such that

\[
a_1v_1 + \cdots + a_nv_n = \vec{0}.
\]
We say that $S$ is linearly independent if it is not linearly dependent.

So, if a set $S$ has redundant vectors, the set $S$ is linearly dependent. From the definition of linear dependence we see that the following is an equivalent way to say that a subset $S$ of vectors from a vector space is linearly independent:

**Proposition 1.2.** Let $V$ be a vector space. A subset $S$ of $V$ is linearly independent if and only if the only linear combination of $S$ giving the zero vector $\vec{0}$ is the trivial linear combination (i.e. $0v_1 + \cdots + 0v_n$).

**Example 2.** In the previous example, we showed that

$$4(1, 3, -4, 2) + -3(2, 2, -4, 0) + 2(1, -3, 2, -4) + 0(-1, 0, 1, 0) = \vec{0}.$$ 

Since at least one of the scalars is nonzero, we know that the set

$$\{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$$

is linearly dependent.

**Example 3.** Determine if the vectors

$$(1, 1, 0), (1, 1, 1), (0, 1, 1)$$

are linearly independent.

To do so, we want to solve the linear system

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
That is, we want to put the following matrix in reduced echelon form

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{bmatrix}
\]


\[
\begin{array}{c}
\rightarrow (-1)R_1 + R_2, \\
\rightarrow (-1)R_2 + R_3, \\
\rightarrow (-1)R_3 + R_1,
\end{array}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

We see from the above that there is a unique solution to this homogeneous system. Namely, the trivial solution \((0, 0, 0)\). In other words the only linear combination of the above vectors that gives the zero vector \(\vec{0}\) is the trivial combination. This means that the above vectors are linearly independent.

**Example 4.** The empty set \(S = \emptyset\) is linearly independent, because linear dependent sets must be non-empty.

**Example 5.** Let \(V\) be a vector space and let \(v \in V\) with \(v \neq \vec{0}\). The set \(S = \{v\}\) is linearly independent. To show this, suppose for the sake of contradiction that \(S\) is linearly dependent. Then there exists a nonzero \(a \in \mathbb{R}\) such that \(av = \vec{0}\). Since \(a \neq 0\), we have

\[
v = \left(\frac{1}{a}\right)v = \frac{1}{a}(av) = \frac{1}{a}\vec{0} = \vec{0},
\]

which contradicts the fact that \(v \neq \vec{0}\). Hence, \(S = \{v\}\) is linearly independent.

Let’s think for a moment about what we’re doing here. To determine if the vectors \(v_1, \ldots, v_n\) are linearly independent, we solved the linear system given by

\[
a_1v_1 + \cdots + a_nv_n = \vec{0}.
\]

That is, the system \(Ax = \vec{0}\) where

\[
A = \begin{bmatrix}
v_1 & \cdots & v_n \\
\end{bmatrix}
\quad \text{and} \quad
x = \begin{bmatrix}
a_1 \\
\vdots \\
a_n
\end{bmatrix}.
\]
This is a homogeneous linear system. Therefore, we know that there is either a unique solution (i.e. the trivial solution) or there are infinitely many solutions. The former would tell us that the vectors $v_1, \ldots, v_n$ are linearly independent, while the latter would imply that there are not.

We know from a previous lecture that if $A$ is a square matrix, then the linear system $Ax = \vec{0}$ has a solution if and only if $A$ is row equivalent to the identity matrix $I$. If this is the case, we know that $\det(A) \neq 0$! Therefore, if we have vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ the matrix $A$ as above is a square matrix, and we can determine if the vectors $v_1, \ldots, v_n$ are linearly independent by computing $\det(A)$. If $\det(A) = 0$, the vectors are linearly dependent. If $\det(A) \neq 0$, the vectors are linearly independent. What happens if $A$ is not square?

**Theorem 1.3.** Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^m$ and set

$$A = \begin{bmatrix} | & | \\ v_1 & \cdots & v_n \end{bmatrix}.$$

Then there are three possibilities:

1. $(n > m)$ The vectors are linearly dependent.

2. $(n = m)$ The vectors are linearly independent if and only if $\det(A) \neq 0$.

3. $(n < m)$ The vectors are linearly independent if and only if there exists an $n \times n$ submatrix $B$ (obtained by deleting $m - n$ rows from $A$) such that $\det(B) \neq 0$.