We investigate approximation properties for $C^*$-algebras and their crossed products by actions and coactions by locally compact groups. We show that Haagerup's approximation constant is preserved for crossed products by arbitrary amenable groups, and we show why this is not always true in the non-amenable case. We also examine similar questions for other forms of the approximation property.

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1. Introduction

A $C^*$-algebra $\mathcal{A}$ has the completely bounded approximation property (CBAP) if there is a net $\{T_\gamma : \mathcal{A} \to \mathcal{A}\}_{\gamma \in \Gamma}$ of finite rank maps, uniformly bounded in the completely bounded norm, which converges in the point norm topology to the identity. The smallest number which can bound such a net is called the Haagerup constant of $\mathcal{A}$, and is denoted by $\Lambda(\mathcal{A})$. If no such net exists we set $\Lambda(\mathcal{A})$ equal to $\infty$. This constant was introduced and studied in a series of papers [4, 5, 9], and is an important isomorphism invariant for $C^*$-algebras. An interesting problem is to determine the functorial properties of the CBAP. In [28], it was shown that $\Lambda$ is multiplicative on the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ of $C^*$-algebras, and $\Lambda$ is invariant under crossed products by discrete amenable groups [29]. The main result of the paper is the extension to the case of general locally compact amenable groups. For von Neumann algebras $\mathcal{M}$ there is a corresponding constant $\Lambda_w(\mathcal{M})$, where point norm convergence is replaced by point $w^*$-convergence. The equality

$$\Lambda_w(\mathcal{M} \times_\alpha G) = \Lambda_w(\mathcal{M})$$

for amenable $W^*$-dynamical systems $(\mathcal{M}, G, \alpha)$ was obtained in [10] and independently in [1]. The difference between norm and $w^*$-convergence means that

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techniques appropriate for \( \Lambda_w \) rarely carry over to \( \Lambda \), and this is the case here. It also seems impossible to extend the methods of [29] to general amenable groups.

Our approach to this problem will make heavy use of the duality theorems of Katayama [13], and of Imai and Takai [12]. This requires us to study coactions, and so we have included in the second section a brief review of those parts of the theory which we will use subsequently. The method which underlies our work is to obtain information about a \( C^* \)-algebra \( A \) by constructing approximate point norm factorizations of \( A \) through a second \( C^* \)-algebra \( B \). There are many instances of this in the literature (see, e.g. [3, 14]) so we have formalized this by calling it the \textit{completely contractive factorization property} (CCFP), defined in the third section. When a pair \((A, B)\) has the CCFP, we show in Theorem 3.1 that slice map properties (see [16, 17, 32]) and various approximation properties, including the CBAP, pass from \( B \) to \( A \). We prove that the pair \((A, A \times_{\alpha, r} G)\) has the CCFP (Theorem 3.2), from which the inequality

\[
\Lambda(A) \leq \Lambda(A \times_{\alpha, r} G)
\]

follows immediately for any locally compact group \( G \). The reverse inequality, for \( G \) amenable, is obtained in the fourth section, by showing that the pair \((A \times_{\alpha, r} G, (A \times_{\alpha, r} G) \times_{\tilde{\alpha}} G)\) has the CCFP for the dual coaction \( \tilde{\alpha} \). Our methods also give preservation of other approximation properties by amenable crossed products by actions and coactions (Theorem 4.6). The last section contains some concluding remarks. We indicate why our results cannot be extended beyond the amenable case, and we take the opportunity to point out that some recent work of Ozawa [23], can be used to settle two open problems from the paper of Haagerup and Kraus [10].

2. Preliminaries

A triple \((A, G, \alpha)\), where \( A \) is a \( C^* \)-algebra, \( G \) is a locally compact group, and \( \alpha : G \to \text{Aut}(A) \) is a homomorphism, is said to be a \( C^* \)-dynamical system if the map \( t \mapsto \alpha_t(a) \) is norm-continuous on \( G \) for each \( a \in A \). The reduced crossed product \( A \times_{\alpha, r} G \) is constructed by taking a faithful representation \( \pi : A \to B(H) \) and associating to it a representation \( \tilde{\pi} : A \to B(L^2(G, H)) \) defined by

\[
(\tilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)
\]

for \( \xi \in L^2(G, H) \). Each \( s \in G \) corresponds to a unitary \( \lambda_s \in B(L^2(G, H)) \) defined by

\[
(\lambda_s \xi)(t) = \xi(s^{-1}t), \quad \xi \in L^2(G, H),
\]

and it is easily checked that

\[
\lambda_s \tilde{\pi}(a) \lambda_s^* = \tilde{\pi}(\alpha_s(a))
\]

for \( s \in G, a \in A \) (the pair \((\tilde{\pi}, \lambda)\) is called covariant). The reduced crossed product is then the norm closure of the span of operators of the form

\[
\int f(s)\tilde{\pi}(a)\lambda_s ds
\]
where \( a \in A \), \( f \in K(G) \), the algebra of continuous functions of compact support on \( G \), and \( ds \) is a fixed choice of left Haar measure on \( G \). This \( C^* \)-algebra is independent of the choice of \( \pi \), since any covariant pair \((\tilde{\pi}, \lambda)\), where \( \pi \) is faithful, induces a faithful representation \( \tilde{\pi} \times \lambda \) of \( \mathcal{A} \times_{\alpha,r} G \) (see [25, Chap. 7]).

We now briefly review the definition of a reduced coaction. A non-degenerate injective *-homomorphism \( \delta \) from \( \mathcal{A} \) into the multiplier algebra \( M(\mathcal{A} \otimes C^*_r(G)) \) of \( \mathcal{A} \otimes C^*_r(G) \) (where \( \otimes \) always denotes the minimal \( C^* \)-tensor product) is called a coaction if it satisfies

\[
(\delta \otimes I)\delta = (I \otimes \delta_G)\delta
\]

where \( \delta_G : C^*_r(G) \rightarrow M(C^*_r(G) \otimes C^*_r(G)) \) is the integrated form of the map \( s \rightarrow s \otimes s \), \( s \in G \); (2.5)

\[
(\delta(\mathcal{A}))(I \otimes C^*_r(G)) \subseteq \mathcal{A} \otimes C^*_r(G),
\]

(2.6)

(see [7, Definition 1.1]). The first condition is called the coaction identity. In the earlier literature, the second condition was called non-degeneracy of the coaction.

A triple \((\mathcal{A}, G, \delta)\), where \( \delta \) is a coaction, is called a cosystem. The crossed product \( \mathcal{A} \times_{\delta} G \) is constructed on \( H \otimes L^2(G) \) in a manner similar to crossed products by actions. For actions the crossed product is generated by a copy of \( \mathcal{A} \) and a copy of \( C^*_r(G) \); for coactions \( C^*_r(G) \) is replaced by \( C^*_0(G) \). The crossed product \( \mathcal{A} \times_{\delta} G \) is the norm closed span of the set of elements

\[
\{ (\pi \otimes I)(\delta(a))(I \otimes M_f) : a \in \mathcal{A}, f \in C^*_0(G) \},
\]

where \( \pi \) is a faithful representation of \( \mathcal{A} \) on \( H \), and \( M_f \) denotes multiplication by \( f \) on \( L^2(G) \) [7, Definition 1.4]. The coaction identity (2.5) and non-degeneracy (2.6) ensure that this is a \( C^* \)-algebra [27]. We also note that the crossed product is independent of the choice of faithful representation. This follows from the work of Raeburn [27], who showed that the full crossed product, defined by universal properties, is isomorphic to the reduced crossed product that we have defined here.

The papers [7, 18, 20, 22, 27] are good references for background material.

Various forms of approximation properties will appear subsequently, so we review them here for the reader’s convenience. We say that \((\mathcal{A}, B, \mathcal{E})\) has the slice map property, where \( \mathcal{E} \) is a closed subspace of a \( C^* \)-algebra \( B \), if any element \( x \in \mathcal{A} \otimes B \), whose right slices by \( \phi \in \mathcal{A}^* \) lie in \( \mathcal{E} \), must be an element of \( \mathcal{A} \otimes \mathcal{E} \). Then \( \mathcal{A} \) has the general slice map property if every such triple \((\mathcal{A}, B, \mathcal{E})\) has the slice map property. If we restrict \( B \) to being the algebra of compact operators \( K(H) \) on a separable Hilbert space, then we say that \( \mathcal{A} \) has the slice map property for subspaces of \( K(H) \).

If there exists a net of finite rank operators \( \{ T_\gamma : \mathcal{A} \rightarrow \mathcal{A} \}_{\gamma \in \Gamma} \) such that \( T_\gamma \otimes I \) converges to \( I \otimes I \) in the point norm topology on \( \mathcal{A} \otimes K(\ell^2) \), then \( \mathcal{A} \) is said to have the operator approximation property (OAP) [6, 17]. If the same conclusion holds true when \( K(\ell^2) \) is replaced by any \( C^* \)-algebra, then we say that \( \mathcal{A} \) has the strong operator approximation property (strong OAP). In general, these two forms of the OAP are distinct, although they coincide for the reduced \( C^* \)-algebras of discrete groups [10].
3. The Completely Contractive Factorization Property

We say that an ordered pair $(A,B)$ of $C^*$-algebras has the completely contractive factorization property (CCFP) if the following condition is satisfied. Given $\varepsilon > 0$, and $a_1, \ldots, a_n \in A$, there exist completely contractive maps $S : A \to B$ and $T : B \to A$ such that

$$
\|T(S(a_i)) - a_i\| < \varepsilon, \quad 1 \leq i \leq n.
$$

(3.1)

Since the set of pairs $(F,\varepsilon)$ of finite subsets $F$ of $A$ and positive numbers $\varepsilon$ can be given a partial order by $(F_1,\varepsilon_1) \leq (F_2,\varepsilon_2)$ if and only if $F_1 \subseteq F_2$ and $\varepsilon_2 \leq \varepsilon_1$, it is clear that our definition is equivalent to the existence of nets $\{S_\gamma : A \to B, T_\gamma : B \to A\}_{\gamma \in \Gamma}$ of complete contractions satisfying

$$
\lim_{\gamma} \|T_\gamma(S_\gamma(a)) - a\| = 0 \quad (3.2)
$$

for all $a \in A$. The point of introducing this concept is to have a simple method of transferring properties from $B$ to $A$. This is formalized in the following result, which will allow us subsequently to concentrate on showing that a particular pair of $C^*$-algebras has the CCPF.

**Theorem 3.1.** Let $(A,B)$ be a pair of $C^*$-algebras having the CCPF.

1. If $C$ is any $C^*$-algebra then the pairs $(A, B \otimes C)$ and $(A \otimes C, B \otimes C)$ have the CCPF.
2. If $(B,C)$ has the CCPF then so too does $(A,C)$.
3. The $C^*$-algebra $A$ inherits each of the following properties from $B$:
   
   - (i) The completely bounded approximation property, and $\Lambda(A) \leq \Lambda(B)$.
   - (ii) Nuclearity.
   - (iii) The general slice map property.
   - (iv) The slice map property for subspaces of $K(H)$.
   - (v) The operator approximation property.
   - (vi) The strong operator approximation property.
   - (vii) Exactness.

**Proof.** (1) Fix an element $c_0 \in C$ of unit norm, and pick a $\phi \in C^*$ such that $\|\phi\| = \phi(c_0) = 1$. If $\{S_\gamma : A \to B, T_\gamma : B \to A\}_{\gamma \in \Gamma}$ are nets of complete contractions such that $\lim_{\gamma} T_\gamma S_\gamma = I$ in the point norm topology, define $S'_\gamma : A \to B \otimes C$, $T'_\gamma : B \otimes C \to A$ by

$$
S'_\gamma(a) = S_\gamma(a) \otimes c_0, \quad T'_\gamma(b \otimes c) = \phi(c)T_\gamma(b),
$$

(3.3)

for $a \in A$, $b \in B$, $c \in C$, $\gamma \in \Gamma$. These maps are complete contractions, and $\lim_{\gamma} T'_\gamma S'_\gamma = I$ in the point norm topology, showing that $(A, B \otimes C)$ has the CCPF.

To show that $(A \otimes C, B \otimes C)$ has the CCPF, just define $S''_\gamma = S_\gamma \otimes I$ and $T''_\gamma = T_\gamma \otimes I$.

(2) This is a simple exercise in the composition of maps.
(3)(i) If $\mathcal{B}$ has the CBAP then there exists a net $\{R_\mu : \mathcal{B} \to \mathcal{B}\}_{\mu \in \mathcal{M}}$ of finite rank maps converging in the point norm topology to $I$ and satisfying $\|R_\mu\|_{cb} \leq \Lambda(\mathcal{B})$. Given $a_1, \ldots, a_n \in \mathcal{A}$ and $\varepsilon > 0$, we can select $\gamma \in \Gamma$ and $\mu \in \mathcal{M}$ such that

$$\|T_\gamma R_\mu S_\gamma(a_i) - a_i\| < \varepsilon, \quad 1 \leq i \leq n,$$

and the result follows, since $\|T_\gamma R_\mu S_\gamma\|_{cb} \leq \Lambda(\mathcal{B})$ and all such maps are finite rank.

(ii) It was shown in [30] that nuclearity of $\mathcal{B}$ is equivalent to having an approximate point norm factorization of $I$ through matrix algebras by complete contractions. This in turn is clearly equivalent to $(\mathcal{B}, K(\ell^2))$ having the CCFP, so the result follows from the transitivity of (2), taking $\mathcal{C}$ to be $K(\ell^2)$.

(iii) Suppose that $\mathcal{B}$ has the general slice map property. Let $\mathcal{C}$ be a $C^*$-algebra with a closed subspace $\mathcal{E}$ and consider an element $x \in \mathcal{A} \otimes \mathcal{C}$, all of whose right slices lie in $\mathcal{E}$. If $\phi \in \mathcal{B}^*$, then the composition of $S_\gamma \otimes I : \mathcal{A} \otimes \mathcal{C} \to \mathcal{B} \otimes \mathcal{C}$ with $R_\phi$ is a right slice map on $\mathcal{A} \otimes \mathcal{C}$. By hypothesis $R_\phi((S_\gamma \otimes I)(x)) \in \mathcal{E}$, so all right slices of $(S_\gamma \otimes I)(x)$ lie in $\mathcal{E}$. Since $\mathcal{B}$ has the general slice map property, $S_\gamma \otimes I(x) \in \mathcal{B} \otimes \mathcal{E}$, so $T_\gamma S_\gamma \otimes I(x) \in \mathcal{A} \otimes \mathcal{E}$ for all $\gamma \in \Gamma$. Take a limit over $\Gamma$ to show that $x \in \mathcal{A} \otimes \mathcal{E}$, proving that $\mathcal{A}$ has the general slice map property.

(iv) This is a special case of (iii).

(v), (vi) We prove only (vi) since the argument also applies to (v). Let $\mathcal{C}$ be any $C^*$-algebra, and suppose that $\mathcal{B}$ has the strong OAP. Given $x_1, \ldots, x_n \in \mathcal{A} \otimes \mathcal{C}$, we may choose $\gamma \in \Gamma$ so that

$$\|T_\gamma S_\gamma \otimes I(x_i) - x_i\| < \varepsilon/2, \quad 1 \leq i \leq n. \quad (3.5)$$

Then we may choose, by hypothesis, a finite rank map $R : \mathcal{B} \to \mathcal{B}$ such that

$$\|R \otimes I(S_\gamma \otimes I)(x_i) - S_\gamma \otimes I(x_i)\| < \varepsilon/2, \quad 1 \leq i \leq n. \quad (3.6)$$

Applying $T_\gamma \otimes I$ to (3.6), and using (3.5) and the triangle inequality, gives

$$\|T_\gamma RS_\gamma \otimes I(x_i) - x_i\| < \varepsilon, \quad 1 \leq i \leq n. \quad (3.7)$$

This shows that $\mathcal{A}$ has the strong OAP.

(vii) Let $\mathcal{J}$ be a norm closed ideal in a $C^*$-algebra $\mathcal{C}$ with quotient map $\pi$, and suppose that $\mathcal{B}$ is exact. To show that $\mathcal{A}$ is exact, we need only prove that the kernel of $I_\mathcal{A} \otimes \pi$ is contained in $\mathcal{A} \otimes \mathcal{J}$ [33]. Consider an element $x \in \ker(I_\mathcal{A} \otimes \pi)$, and observe that $S_\gamma \otimes I(x) \in \ker(I_\mathcal{B} \otimes \pi)$, which is $\mathcal{B} \otimes \mathcal{J}$ by exactness of $\mathcal{B}$. Apply $T_\gamma \otimes I_\mathcal{C}$ and take a limit over $\gamma \in \Gamma$ to obtain $x \in \mathcal{A} \otimes \mathcal{J}$. This shows that $\mathcal{A}$ is exact.

We close this section by exhibiting one pair of $C^*$-algebras with the CCFP.

**Theorem 3.2.** Let $G$ be a locally compact group and let $\alpha : G \to \text{Aut}(\mathcal{A})$ be a strongly continuous action on a $C^*$-algebra $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{A} \rtimes_{\alpha, r} G)$ has the CCFP.
Proof. Consider \(a_1, \ldots, a_n \in \mathcal{A}\) and \(\varepsilon > 0\). We fix a positive number \(\varepsilon'\) to be chosen later. For each \(f \in K(G)\), \(a \in \mathcal{A}\), let \(f \cdot a\) denote the element of \(K(G, \mathcal{A})\) whose value at \(s \in G\) is \(f(s)a\). Then define \(S_f : \mathcal{A} \rightarrow \mathcal{A} \times_{\alpha,r} G\) by

\[
S_f(a) = (\tilde{\pi} \times \lambda)(f \cdot a), \quad a \in \mathcal{A}. \tag{3.8}
\]

Then

\[
S_f(a) = \int f(s)\tilde{\pi}(a)\lambda_s ds. \tag{3.9}
\]

Each map \(a \rightarrow \tilde{\pi}(a)\lambda_s\) is a complete contraction, so \(S_f\) may be viewed as a vector integral of such maps, giving \(\|S_f\|_{cb} \leq \|f\|_1\).

For each \(\xi \in L^2(G) \cap K(G)\), let \(\omega_\xi\) be the associated normal vector functional on \(B(L^2(G))\). Then the left slice map \(L_{\omega_\xi}\) is well defined on \(B(H) \otimes B(L^2(G))\) (which we identify with \(B(L^2(G, H))\), and \(\|L_{\omega_\xi}\|_{cb} = \|\xi\|_2^2\). Let \(\tilde{T}_\xi\) be the restriction of \(L_{\omega_\xi}\) to \(\mathcal{A} \times_{\alpha,r} G\). Then \(\|\tilde{T}_\xi\|_{cb} \leq \|\xi\|_2^2\). We first show that the range of \(\tilde{T}_\xi\) is contained in \(\pi(\mathcal{A})\). In the following calculation, our assumptions on continuity of \(\xi\) and \(f\) will automatically imply that the integrands are integrable. If \(h, k \in H\), then

\[
\langle \tilde{T}_\xi(\tilde{\pi} \times \lambda)(f \cdot a)h, k \rangle = \langle (\tilde{\pi} \times \lambda)(f \cdot a)h \otimes \xi, k \otimes \xi \rangle
\]

\[
= \int \int \langle f(s)\tilde{\pi}(a)\lambda_s(h \otimes \xi)(t), k \otimes \xi(t) \rangle dt ds
\]

\[
= \int f(s)\xi(s^{-1}t)\overline{\xi(t)}\langle \pi(\alpha_t^{-1}(a)h), k \rangle dt ds
\]

\[
= \int f \ast \xi(t)\overline{\xi(t)}\langle \pi(\alpha_t^{-1}(a)h), k \rangle dt. \tag{3.10}
\]

It follows from (3.10) that

\[
\tilde{T}_\xi(\tilde{\pi} \times \lambda)(f \cdot a) = \int f \ast \xi(t)\overline{\xi(t)}\pi(\alpha_t^{-1}(a)) dt \tag{3.11}
\]

and this last integral is an element of \(\pi(\mathcal{A})\). If we let \(T_\xi\) denote \(\pi^{-1}\tilde{T}_\xi\), then we have shown that \(\|T_\xi\|_{cb} \leq \|\xi\|_2^2\), and \(T_\xi\) maps \(\mathcal{A} \times_{\alpha,r} G\) into \(\mathcal{A}\), since the span of the elements \(\tilde{\pi} \times \lambda(f \cdot a)\), \(f \in K(G)\), \(a \in \mathcal{A}\), is norm dense in \(\mathcal{A} \times_{\alpha,r} G\).

For the given elements \(a_1, \ldots, a_n\) we now wish to choose \(f\) and \(\xi\) so that

\[
\|T_\xi S_f(a_i) - a_i\| < \varepsilon, \quad 1 \leq i \leq n. \tag{3.12}
\]

We restrict attention to \(f \in K(G)^+, \|f\|_1 = 1\), and \(\xi \in L^2(G)^+, \|\xi\|_2 = 1\), so that we already have \(\|T_\xi\|_{cb}, \|S_f\|_{cb} \leq 1\). For each \(i\), the map \(t \rightarrow \alpha_t(a_i)\) is continuous on \(G\), so we may choose a symmetric neighborhood \(U\) of \(e \in G\) such that

\[
\|\alpha_t(a_i) - a_i\| < \varepsilon', \quad 1 \leq i \leq n, \quad t \in U. \tag{3.13}
\]

Now choose a non-negative \(\xi \in L^2(G) \cap K(G)\), \(\|\xi\|_2 = 1\), whose support is contained in \(U\), and let \(\xi_s\) denote the left translate \(\xi(s^{-1}t)\) of \(\xi\). The map \(s \rightarrow \|\xi - s\xi\|_2\) is continuous on \(G\), by [11, Theorem 20.4], so there is a neighborhood \(V\) of \(e\), contained
in $U$, within which $\|\xi - s\xi\|_2 < \epsilon'$. In particular, the Cauchy–Schwarz inequality shows that

$$|\langle s\xi - \xi, \xi \rangle| < \epsilon', \quad s \in V.$$  \hfill (3.14)

Finally we choose $f \in K(G)^+, \|f\|_1 = 1$, and having support in $V$. We are now ready to show that, with these choices, $\tilde{T}_\xi S_f(a_i)$ is close to $\pi(a_i)$ for $1 \leq i \leq n$.

Let $h, k \in H$ be arbitrary vectors of unit norm. Then

$$\|\langle \tilde{T}_\xi S_f(a_i) - \pi(a_i) \rangle h, k \| = \left| \iint f(s)\xi(s^{-1}t)\xi(t)\langle \pi(\alpha_{t^{-1}}(a_i))h, k \rangle dsdt - \langle \pi(a_i)h, k \rangle \right|$$

$$\leq \left| \iint f(s)\xi(s^{-1}t)\xi(t)\langle \pi(\alpha_i)h, k \rangle dsdt - \langle \pi(a_i)h, k \rangle \right|$$

$$+ \left| \iint f(s)\xi(s^{-1}t)\xi(t)(\pi(\alpha_{t^{-1}}(a_i)) - \pi(a_i))h, k \rangle dsdt \right|.$$  \hfill (3.15)

We now estimate these integrals separately. Since

$$\iint f(s)\xi(t)^2 dsdt = 1,$$  \hfill (3.16)

the first may be rewritten as

$$\left| \iint \langle \pi(\alpha_i)h, k \rangle f(s)\xi(t)(s\xi(t) - \xi(t)) dsdt \right| \leq \int_V \|a_i\| f(s) |\langle s\xi - \xi, \xi \rangle| ds$$

$$\leq \int \epsilon' \|a_i\| f(s) ds$$

$$= \epsilon' \|a_i\|.$$  \hfill (3.17)

Here we have used (3.14) and Fubini’s theorem, which is permissible because the integrand is a continuous function of compact support on $G \times G$.

For the second integral, we change the order of integration, and observe that the $t$-variable can be restricted to $U$ (because of the $\xi(t)$ term). This yields

$$\left| \iint f(s)\xi(s^{-1}t)\xi(t)(\pi(\alpha_{t^{-1}}(a_i)) - \pi(a_i))h, k \rangle dsdt \right|$$

$$\leq \iint f(s)\xi(s^{-1}t)\xi(t)\|\alpha_{t^{-1}}(a_i) - a_i\| dt\| ds$$

$$\leq \iint \epsilon' f(s)\xi(s^{-1}t)\xi(t) dt\| ds$$

$$= \int \epsilon' f(s) \langle s\xi - \xi, \xi \rangle + \langle \xi, \xi \rangle ds$$

$$\leq (\epsilon')^2 + \epsilon'.$$  \hfill (3.18)
again using (3.14). Returning to (3.15), the two estimates (3.17) and (3.18) lead to
\[
|\langle T_\epsilon S_f(a_i) - \pi(a_i)h, k \rangle| \leq \varepsilon'\|a_i\| + \varepsilon' + (\varepsilon')^2.
\] (3.19)
The unit vectors were arbitrary in (3.19), so
\[
\left| T_\epsilon S_f(a_i) - a_i \right| \leq \varepsilon'(1 + \|a_i\| + \varepsilon').
\] (3.20)
The proof is completed by choosing \( \varepsilon' \) so small that
\[
\varepsilon'(1 + \varepsilon' + \max_i \|a_i\|) < \varepsilon.
\] (3.21)
Thus \((A, A \times_{\alpha, r} G)\) has the CCFP.

**Corollary 3.3.** For any C*-dynamical system \((A, G, \alpha)\),
\[
\Lambda(A) \leq \Lambda(A \times_{\alpha, r} G).
\] (3.22)

**Proof.** Apply Theorem 3.1 (3)(i) to the pair \((A, A \times_{\alpha, r} G)\), which has the CCFP by Theorem 3.2.

In [29], the equality \(\Lambda(A) = \Lambda(A \times_{\alpha, r} G)\) was proved when \(G\) was amenable and discrete, or Abelian and compact. Theorem 3.2 already allows us to improve the situation.

**Corollary 3.4.** Let \(G\) be an Abelian locally compact group, and let \(\alpha\) be an action on a C*-algebra \(A\). Then
\[
\Lambda(A) = \Lambda(A \times_{\alpha, r} G).
\] (3.23)

**Proof.** Since \(G\) is Abelian, the Takai duality theorem [31], states that there is a dual action \(\hat{\alpha}\) of \(\hat{G}\) on \(A \times_{\alpha, r} G\), and \((A \times_{\alpha, r} G) \times \hat{\alpha}, \hat{G}\) is isomorphic to \(A \otimes K(L^2(G))\). Two applications of Corollary 3.3 give
\[
\Lambda(A) \leq \Lambda(A \times_{\alpha, r} G) \leq \Lambda((A \times_{\alpha, r} G) \times \hat{\alpha}, \hat{G})
\]
\[
= \Lambda(A \otimes K(L^2(G))
\]
\[
= \Lambda(A),
\] (3.24)
the last equality being a special case of [28, Theorem 2.2].

4. Coactions

In this section we investigate the counterparts of the results of the previous section for coactions of groups on C*-algebras. One might hope that the pair \((A, A \times_{\alpha} G)\) has the CCFP, but this is not possible in general (we return to this point later). However, if \(A\) is replaced by a reduced crossed product \(A \times_{\alpha, r} G\), the dual coaction \(\hat{\alpha}\) is easy to describe, and a concrete faithful representation of \((A \times_{\alpha, r} G) \times \hat{\alpha} G\) can be given on the Hilbert space \(H \otimes L^2(G) \otimes L^2(G)\), when \(A\) is faithfully represented on \(H\) by \(\pi\),
Since this restriction. The algebra \((A \times_{\alpha, r} G) \times_{\tilde{\alpha}} G\) is generated by a copy of \(A \times_{\alpha, r} G\) and a copy of \(C_0(G)\) as multiplication operators on the second copy of \(L^2(G)\). Then, for \(a \in A, h_1, h_2 \in H, \xi_1, \xi_2, \eta_1, \eta_2 \in L^2(G) \cap K(G), f, g \in K(G)\), a routine calculation shows that

\[
\langle \tilde{\alpha}(\pi \times \lambda(f \cdot a))I \otimes I \otimes M_g(h_1 \otimes \xi_1 \otimes \eta_1), h_2 \otimes \xi_2 \otimes \eta_2 \rangle
= \iiint f(s)\langle \pi(\alpha_{r-1}(a))h_1, h_2\rangle \xi_1(s^{-1}r)\eta_1(s^{-1}t)g(s^{-1}t)\xi_2(r)\eta_2(t)drdsdt.
\]  

(4.1)

Our requirement that all the functions lie in \(K(G)\) means that we need not concern ourselves with the order of integration here, or subsequently.

**Theorem 4.1.** Let \(G\) be an amenable locally compact group, let \((A, G, \alpha)\) be a \(C^\ast\)-dynamical system, and let \(\tilde{\alpha}\) be the dual coaction of \(G\) on \(A \times_{\alpha, r} G\). Then the pair \((A \times_{\alpha, r} G, (A \times_{\alpha, r} G) \times_{\tilde{\alpha}} G)\) has the CCFP.

**Proof.** It is clearly sufficient to consider a finite number of elements \(\tilde{\pi} \times \lambda(f_i \cdot a_i) \in A \times_{\alpha, r} G, 1 \leq i \leq n\), where \(f_i \in K(G)\) and \(a_i \in A, \|f_i\|_1 = 1, \|a_i\| = 1\). For \(g \in K(G)\), define \(S_g : A \times_{\alpha, r} G \rightarrow (A \times_{\alpha, r} G) \times_{\tilde{\alpha}} G\) by

\[
S_g(x) = \tilde{\alpha}(x)I \otimes I \otimes M_g, \ x \in A \times_{\alpha, r} G.
\]  

(4.2)

Since \(\tilde{\alpha}\) is a homomorphism, it is clear that \(\|S_g\|_{cb} \leq \|g\|_{\infty}\). For each \(\eta \in L^2(G) \cap K(G)\), we define a map \(T_\eta : (A \times_{\alpha, r} G) \times_{\tilde{\alpha}} G \rightarrow A \times_{\alpha, r} G, \|T_\eta\|_{cb} \leq \|\eta\|_2^2\), by slicing in the second copy of \(L^2(G)\) by the vector functional \(\omega_\eta\). *A priori, \(T_\eta\) maps into \(B(H \otimes L^2(G))\), but it will become apparent from the subsequent calculations that the range of \(T_\eta\) lies in \(A \times_{\alpha, r} G\). From (4.1),

\[
\langle T_\eta(S_g(\tilde{\pi} \times \lambda(f \cdot a)))h_1 \otimes \xi_1, h_2 \otimes \xi_2 \rangle
= \iiint f(s)\langle \pi(\alpha_{r-1}(a))h_1, h_2\rangle \xi_1(s^{-1}r)\xi_2(r)\eta(s^{-1}t)g(s^{-1}t)\eta(t)drdsdt. 
\]  

(4.3)

Let \(F_{\eta, g} \in K(G)\) be defined by

\[
F_{\eta, g}(s) = \int \eta(s^{-1}t)g(s^{-1}t)\eta(t)dt, \ s \in G.
\]  

(4.4)

Then we may integrate first with respect to \(t\) in (4.3) to conclude that

\[
T_\eta(S_g(\tilde{\pi} \times \lambda(f \cdot a))) = \tilde{\pi} \times \lambda((F_{\eta, g}f) \cdot a).
\]  

(4.5)

This establishes that each \(T_\eta\) maps into \(A \times_{\alpha, r} G\). It also follows from (4.5) that

\[
T_\eta(S_g(\tilde{\pi} \times \lambda(f_i \cdot a_i))) - \tilde{\pi} \times \lambda(f_i \cdot a_i) = \tilde{\pi} \times \lambda((F_{\eta, g} - 1)f_i \cdot a_i),
\]  

(4.6)
so to show that $T_{\eta}S_g$ is approximately the identity on the elements $\tilde{\pi} \times \lambda(f_i \cdot a_i)$, it suffices to choose $\eta$ and $g$ so that the right-hand side of (4.6) is small in norm. A simple estimate gives
\[
\|\tilde{\pi} \times \lambda((F_{\eta,g} - 1)f_i \cdot a_i)\| \leq \|(F_{\eta,g} - 1)f_i\|_1,
\]
since $\|a_i\| = 1$, so given $\varepsilon > 0$, it suffices to find $\eta \in L^2(G) \cap K(G)$, $\|\eta\|_2 \leq 1$, $g \in C_0(G)$, $\|g\|_\infty \leq 1$, such that
\[
|F_{\eta,g} - 1| < \varepsilon
\]
on the combined supports of the $f_i$’s.

We now use the hypothesis that $G$ is amenable. Let $E_1$, a compact subset of $G$, be the union of the supports of the $f_i$’s. By [25, Proposition 7.3.8], there is a unit vector $\eta \in L^2(G)$ (which we may take to be in $L^2(G) \cap K(G)$) such that
\[
\eta * \tilde{\eta}(s) = \int \eta(t)\tilde{\eta}(s^{-1}t)dt
\]
satisfies
\[
|\eta * \tilde{\eta}(s) - 1| < \varepsilon, \quad s \in E_1.
\]
Here $\tilde{\eta}$ is defined by $\tilde{\eta}(s) = \overline{\eta(s^{-1})}$. Let $E_2$ be the support of $\eta$, and select $g \in K(G)$, $\|g\|_\infty = 1$, and $g \equiv 1$ on $E_2$. For these choices, it follows from (4.4), (4.9) and (4.10) that
\[
|F_{\eta,g}(s) - 1| < \varepsilon, \quad s \in E_1.
\]
An immediate consequence of (4.11) is
\[
\|(F_{\eta,g} - 1)f_i\|_1 < \varepsilon, \quad 1 \leq i \leq n,
\]
completing the proof.

We can now state one of our main results.

Corollary 4.2. If $G$ is an amenable group and $(\mathcal{A}, G, \alpha)$ is a $C^*$-dynamical system, then
\[
\Lambda(\mathcal{A}) = \Lambda(\mathcal{A} \rtimes_{\alpha,r} G).
\]

Proof. We have already shown that $\Lambda(\mathcal{A}) \leq \Lambda(\mathcal{A} \rtimes_{\alpha,r} G)$. For the converse, let $\hat{\alpha}$ be the dual coaction. By Theorem 4.1, the pair $(\mathcal{A} \rtimes_{\alpha,r} G, (\mathcal{A} \rtimes_{\alpha,r} G) \rtimes \hat{\alpha} G)$ has the CCFP, so by Theorem 3.1 (3)(i)
\[
\Lambda(\mathcal{A} \rtimes_{\alpha,r} G) \leq \Lambda((\mathcal{A} \rtimes_{\alpha,r} G) \rtimes \hat{\alpha} G).
\]
But $(\mathcal{A} \rtimes_{\alpha,r} G) \rtimes \hat{\alpha} G$ is isomorphic to $\mathcal{A} \otimes K(L^2(G))$, [12], so (4.14) becomes
\[
\Lambda(\mathcal{A} \rtimes_{\alpha,r} G) \leq \Lambda(\mathcal{A} \otimes K(L^2(G))) = \Lambda(\mathcal{A})
\]
by [28, Theorem 2.2], proving (4.13).
We may now use non-Abelian duality to obtain the counterpart of Corollary 4.2 for coactions. Curiously, it does not seem possible to prove this without first considering the special case of Theorem 4.1. A similar situation arose in the fourth section of [7].

**Corollary 4.3.** Let $\delta$ be a non-degenerate coaction of an amenable group $G$ on a $C^*$-algebra $A$. Then

$$\Lambda(A \times_\delta G) = \Lambda(A).$$

(4.16)

**Proof.** By [13], there is a dual action $\hat{\delta}$ of $G$ on $A \times_\delta G$ such that $(A \times_\delta G) \times_{\hat{\delta}, r} G$ is isomorphic to $A \otimes K(L^2(G))$. Using Corollary 4.2, with $A \times_\delta G$ in place of $A$, we see that

$$\Lambda(A \times_\delta G) = \Lambda((A \times_\delta G) \times_{\hat{\delta}, r} G) = \Lambda(A \otimes K(L^2(G))) = \Lambda(A),$$

(4.17)

proving (4.16).

**Remark 4.4.** If $G$ is non-amenable then (4.16) may fail, but the inequality

$$\Lambda(A \times_\delta G) \leq \Lambda(A)$$

(4.18)

is an immediate consequence of applying our duality methods and Corollary 3.3.

**Remark 4.5.** Corollary 4.2 also holds true for the twisted crossed products of [24]. When $G$ is amenable, full and reduced crossed products coincide, so from [24] a twisted crossed product of $A$ by $G$ is stably isomorphic to a crossed product of $A$ by $G$. Our assertion then follows from Corollary 4.2, since tensoring by the algebra of compact operators does not change $\Lambda(\cdot)$.

We conclude this section by investigating whether actions and coactions preserve the properties stated in Theorem 3.1.

**Theorem 4.6.** Let $G$ be an amenable group and let $\alpha, \delta$ be respectively an action and a coaction of $G$ on a $C^*$-algebra $A$. For any one of the properties (i)–(vii) of Theorem 3.1, all three $C^*$-algebras $A, A \times_{\alpha, r} G$, and $A \times_\delta G$ have this property or all three do not.

**Proof.** Since the pair $(A, A \otimes K(H))$ has the CCFP for any Hilbert space $H$ (Theorem 3.1 (1)), these properties all transfer from $A \otimes K(H)$ to $A$. The verification that these properties all transfer from $A$ to $A \otimes K(H)$ is essentially routine, based on the nuclearity of $K(H)$. To give the flavor, we will prove this for (iii), and leave the others to the reader.

Suppose that $A$ has the general slice map property and fix a $C^*$-algebra $B$ with a closed subspace $\mathcal{E}$. Let $x \in (A \otimes K(H)) \otimes B$ be an element whose right slices lie in $\mathcal{E}$. If $\phi \in A^*$ and $\psi \in K(H)^*$, then slicing by $\phi \otimes \psi$ is the same as slicing by $\phi$ and then by $\psi$. Thus the element $y \in K(H) \otimes B$ obtained from $x$ by slicing by $\phi$
has the property that all right slices are in $E$. Since $K(H)$ has the general slice map property, by nuclearity, it follows that $y \in K(H) \otimes E$. Since $\phi \in \mathcal{A}^*$ was arbitrary, the general slice map property for $\mathcal{A}$ implies that $x \in \mathcal{A} \otimes K(H) \otimes E$ as required.

We have already shown in Theorems 3.2 and 4.1 that the pairs $(\mathcal{A}, \mathcal{A} \times_{\alpha, r} G)$ and $(\mathcal{A} \times_{\alpha, r} G, \mathcal{A} \otimes K(L^2(G))$ have the CCFP, so we conclude from Theorem 3.1 and the preceding remarks that these properties transfer between $\mathcal{A}$ and $\mathcal{A} \times_{\alpha, r} G$ in both directions. Applying this to the pair $\mathcal{A} \times_{\delta} G$ and $(\mathcal{A} \times_{\delta} G) \times_{\delta, r} G \approx \mathcal{A} \otimes K(L^2(G))$, we see that they also transfer in both directions between $\mathcal{A}$ and $\mathcal{A} \times_{\delta} G$.

\textbf{Remark 4.7.} We note that (ii) in Theorem 4.6 gives a new method of showing the well known result (see [8, 19, 27]) that nuclearity is preserved by actions and coactions of amenable groups. The same is also true for exactness, by (vii), where the original proofs of the action and coaction cases are due respectively to Kirchberg and to Ng [15, 21].

5. Concluding Remarks

If $\alpha$ is the trivial action of $G$ on $\mathcal{A}$, then the crossed product $\mathcal{A} \times_{\alpha, r} G$ is isomorphic to $\mathcal{A} \otimes C^*_r(G)$. It then follows from [28, Theorem 2.2] that $\Lambda(\mathcal{A} \times_{\alpha, r} G) = \Lambda(\mathcal{A})\Lambda(C^*_r(G))$, where both are discrete. There is a naturally induced action $\alpha$ of $G$ on $C^*_r(H)$, and the crossed product $C^*_r(H) \times_{\alpha, r} G$ is isomorphic to the reduced $C^*$-algebra $C^*_r(H \times_{\rho} G)$ of the semi-direct product $H \times_{\rho} G$ [2]. It was established in [9] that, for the natural action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$, $\Lambda(C^*_r(\mathbb{Z}^2 \times_{\rho} SL(2, \mathbb{Z}))) = \infty$, while $\Lambda(C^*_r(\mathbb{Z}^2))$ and $\Lambda(C^*_r(SL(2, \mathbb{Z})))$ are both finite. Thus we cannot expect an upper estimate for $\Lambda(\mathcal{A} \times_{\alpha, r} G)$ in terms of $\Lambda(\mathcal{A})$ and $\Lambda(C^*_r(G))$.

Any group $G$ acts trivially on $\mathbb{C}$, and the resulting reduced crossed product is $C^*_r(G)$. By duality, $C^*_r(G) \times_{\delta} G$ (where $\delta$ is the dual coaction) is isomorphic to $K(H)$. Thus the pair $(C^*_r(G), C^*_r(G) \times_{\delta} G)$ can only have the CCFP when $C^*_r(G)$ is nuclear, confirming our statement at the beginning of the last section that pairs $(\mathcal{A}, \mathcal{A} \times_{\delta} G)$ will not always have this property.

In [10], Haagerup and Kraus introduced the approximation property (AP) for groups (we refer to this paper for the definition which will not be needed here). For a discrete group $\Gamma$ [10, Theorem 2.1] shows that the AP for $\Gamma$ is equivalent to the strong OAP for $C^*_r(\Gamma)$. They left open the question of whether the AP passes to quotients by normal subgroups, but expressed the view that this was unlikely since every countable discrete group is a quotient of $F_{\infty}$, which does have the AP [5]. Since the strong OAP implies exactness (remarks preceding [10, Theorem 2.2]), the work of Ozawa [23], showing the existence of discrete groups $\Gamma$ for which $C^*_r(\Gamma)$
is not exact, also provides examples where the AP fails. The above discussion then shows that there are normal subgroups \( N \) of \( F_\infty \) such that \( C^*_r(F_\infty) \) and \( C^*_r(N) \) have the strong OAP while the \( C^* \)-algebras \( C^*_r(F_\infty/N) \) do not.

Although we have not needed to do so, we could have weakened the definition of the CCFP by requiring that the nets of maps be uniformly bounded in the completely bounded norm. The completely bounded factorization property (CBFP) would be an appropriate name for this potentially useful concept. Most of the statements and proofs of Theorems 3.1 and 4.6 are valid with little change, although for nuclearity a result from [26] (the remark preceding Theorem 2.10) is required: \( \mathcal{A} \) is nuclear if and only if \( (\mathcal{A}, K(\mathcal{H})) \) has the CBFP.

References

9. U. Haagerup, Group \( C^* \)-algebras without the completely bounded approximation property, unpublished manuscript.