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Three-dimensional swirling flows in a tall cylinder driven by a rotating endwall

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The onset and nonlinear dynamics of swirling flows in relatively tall cylinders driven by the rotation of an endwall are studied numerically. These flows are distinguished from the more widely studied swirling flows in shorter cylinders; the instability in the taller cylinders is direct to three-dimensional flows rather than to unsteady axisymmetric flows. The simulations are in very good agreement with recent experiments in terms of the critical Reynolds number, frequency, and azimuthal wavenumber of the flows, but there is disagreement in the interpretation of these flows. We show that these flows are indeed rotating waves and that they have the same vorticity distributions as the flows measured using particle image velocimetry in the experiments. Identifying these as rotating waves gives a direct connection with prior linear stability analysis and the three-dimensional flows found in shorter cylinders as secondary instabilities leading to modulated rotating waves.

I. INTRODUCTION

There has been much recent interest in swirling flows in an enclosed cylinder driven by a rotating endwall when the height-to-radius aspect ratio is greater than about 3, as the onset of instability is to three-dimensional flow when the endwall rotation exceeds a critical value. The most recent of these studies is curious as it analyzes particle image velocimetry (PIV) data of these swirling flows from a multiple-helix point of view and makes the claim that the experimentally observed flows for cylinder aspect ratios greater than about 3 are different from the three-dimensional rotating wave states found at lower aspect ratios. If this is indeed the case, then it begs the question as to what are these new states and how do they come about?

The experiments report that the new states appear directly from the steady axisymmetric basic state as the endwall rotation, non-dimensionalized as a Reynolds number, is increased beyond a critical value dependent on the aspect ratio. In the more recent paper, they analyze their experimental data in terms of helical modes. Their dye and air-bubble visualizations show very distinct helical concentrations of either dye or air bubbles (they have conducted two sets of experiments with different apparatus, working fluids, and measurement and visualization techniques, and have obtained consistent results), and hence their suggestion that these states are different from the rotating wave states that bifurcate from the basic state. In an earlier paper, reporting on very similar experiments they note a discrepancy between the experimentally observed critical Reynolds number and the critical Reynolds number determined from the linear stability analysis for the corresponding azimuthal wavenumber. In that earlier paper, they suggested that this may be due to a subcritical bifurcation to three-dimensional flow, but have yet to investigate if this is the case. Up to now, no nonlinear simulations have been reported to help address this issue. Our simulations reported here show that the onset of three-dimensional flow is supercritical and that there is no hysteresis.

Equivariant dynamical systems theory states that when an $SO(2)$ axisymmetric steady state loses stability to three-dimensional perturbations as a single parameter is varied, and that if
the bifurcation is supercritical (energy of the three-dimensional perturbation growing linearly with
the bifurcation parameter and the frequency varying very weakly with the bifurcation parameter)
then the bifurcating state is a rotating wave. Rotating waves are three-dimensional structures that
are invariant in an appropriately rotating frame of reference (the types of states the authors claim
are not what they observe).

To date, the only nonlinear three-dimensional simulations of the flows in tall cylinders are
for aspect ratio $H/R = 4.0$ and the results are not consistent with the experimental observations. The
main discrepancy is that the numerical simulations first show an onset of axisymmetric time-
periodic flow undergoing a secondary instability to three-dimensional flow, whereas the experi-
ments report a direct transition to three-dimensional flow from the steady axisymmetric basic state
at a much lower $Re$. Our simulations presented here agree with the experiments in terms of the
critical $Re$, frequency, and azimuthal wave number at onset.

In this paper we address these issues, making direct comparisons with the PIV and laser
Doppler anemometry (LDA) data provided from the experiments and the corresponding linear sta-
bility analysis, and also relate our large aspect ratio results to lower aspect ratio experiments
and nonlinear simulations, as well as linear stability analysis, in order to place the tall cylinder
results in a proper context with regard to the results for shorter cylinders.

II. GOVERNING EQUATIONS AND NUMERICAL METHODS

Consider the flow in a stationary circular cylinder of radius $R$ and height $H$, completely filled
with a fluid of kinematic viscosity $\nu$. The flow is driven by the rotation of the bottom endwall with
an angular speed $\Omega$. A schematic of the flow system is shown in Fig. 1. The Navier–Stokes equa-
tions, non-dimensionalized using $R$ as the length scale and $1/\Omega$ as the time scale, are

$$ (\partial_t + u \cdot \nabla)u = -\nabla p + 1/Re \nabla^2 u, \quad \nabla \cdot u = 0, $$

where $u = (u,v,w)$ is the velocity field in polar coordinates $(r,\theta,z) \in [0,1] \times [0,2\pi] \times [0,H/R]$ and $p$
is the kinematic pressure. The corresponding vorticity field is $\nabla \times u = (\xi,\eta,\zeta) = (1/r \partial_\theta w - \partial_z v, \partial_z u - \partial_r w, 1/r \partial_r (rv) - 1/r \partial_\theta u)$.

There are two governing parameters: the aspect ratio $H/R$ and the Reynolds number
$Re = \Omega R^2/\nu$. The boundary conditions are no-slip: at the stationary top, $z = H/R$, and at the station-
ary sidewall, $r = 1$, $(u,v,w) = (0,0,0)$, while at the rotating bottom wall, $z = 0$, $(u,v,w) = (0,r,0)$. The
results presented here are for $Re \in [2000, 3200]$ and $H/R \in [3.5, 5.3]$.

![FIG. 1. Schematic of the apparatus.](image)
The governing equations and boundary conditions are invariant under rotations through arbitrary angle $\phi$ about the axis, $R_\phi$, whose action on any solution $u = (u,v,w)(r,\theta,z,t)$ is

$$R_\phi(u,v,w)(r,\theta,z,t) = (u,v,w)(r,\theta+\phi,z,t);$$

i.e., the symmetry group of the system is $SO(2)$. They are also invariant under arbitrary translations in time $\tau$, whose action is

$$\Phi_\tau(u,v,w)(r,\theta,z,t) = (u,v,w)(r,\theta,z,t+\tau).$$

Hence, the basic states, which depend on the two parameters $Re$ and $H/R$, are steady and axisymmetric.

The governing equations (1) have been solved using a second order time-splitting method, with space discretized via a Galerkin–Fourier expansion in axisymmetric. Hence, the basic states, which depend on the two parameters $Re$ and $H/R$, are steady and axisymmetric.

The spectral solver$^{16}$ has recently been tested and used in a wide variety of enclosed cylinder flows.$^{17-20}$ For the solutions presented here, we have used $n_r = 48$, $n_z = 96$ Chebyshev modes in the radial and axial directions and $n_\theta = 13$ Fourier modes in the azimuthal direction. The time-step used was $\delta t = 0.005$.

III. RESULTS

We shall focus our efforts on 4 aspect ratios, three of these ($H/R = 3.5$, 4.6, and 5.3) correspond to those for which the experiments have focused on,$^6,^7$ and detailed onset experimental data are available,$^5$ and the fourth, $H/R = 4.0$, is the case where there are discrepancies between the experiments and numerical simulations.$^2$ Table I summarizes our numerical results and the experimental results$^5$ for these 4 values of $H/R$, giving the azimuthal wave number, $m$, of the bifurcating state, the critical $Re$ for onset (the experiments were conducted at discrete values of $Re$, using steps of $\Delta Re = 100$, so in the table we present a bracket bounded by the highest $Re$ for sustained axisymmetric flow and the lowest $Re$ for sustained three-dimensional flow at the given $H/R$, as reported), and the frequency of the flow oscillations at a point in space (numerically, we obtain this frequency from a time series of the axial velocity at $(r,\theta,z) = (0.5,0.0,0.5H/R)$, whereas the experiments$^21$ determined the frequency from laser Doppler anemometry measurements of axial and azimuthal velocities at a number of points). Along with the experimental results, they$^5$ also determined the critical $Re$ from numerical linear stability analysis, and they note that the experiments predict slightly lower critical $Re$ for the $m = 3$ and $m = 2$ branches, but higher critical $Re$ for the $m = 4$ branch. The critical $Re$ from their linear stability analysis is in better agreement with our nonlinear numerical estimates; but in any case, all three are in very close agreement. They speculated that the slight disagreement between their experimental and theoretical estimates may be due to subcritical bifurcations (which they did not investigate).

Our numerical estimates of the critical $Re$ reported in Table I were determined from the energy in the non-axisymmetric components of the flows. This was measured using the $L_2$-norms of the azimuthal Fourier modes of a given solution,

$$E_m = \frac{1}{2} \int_{z=0}^{z=H/R} \int_{r=0}^{r=1} u_m \cdot u_m^* \, r \, dr \, dz,$$

TABLE I. Critical $Re$, frequencies $\omega$ and azimuthal wavenumber $m$ at various $H/R$ as determined by our nonlinear simulations and reported from experiments.$^5$

<table>
<thead>
<tr>
<th>$H/R$</th>
<th>$m$</th>
<th>Numerical $Re_c$</th>
<th>Numerical $\omega$</th>
<th>Experimental $Re_c$</th>
<th>Experimental $\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>3</td>
<td>2131.2</td>
<td>0.2967</td>
<td>(2000, 2100)</td>
<td>0.29</td>
</tr>
<tr>
<td>4.0</td>
<td>3</td>
<td>2433.9</td>
<td>0.2680</td>
<td>(2200, 2300)</td>
<td>0.27</td>
</tr>
<tr>
<td>4.6</td>
<td>2</td>
<td>2834.3</td>
<td>0.1350</td>
<td>(2600, 2700)</td>
<td>0.14</td>
</tr>
<tr>
<td>5.3</td>
<td>4</td>
<td>3035.6</td>
<td>0.4009</td>
<td>(3000, 3100)</td>
<td>0.38</td>
</tr>
</tbody>
</table>
where $u_m$ is the $m$th Fourier mode of the velocity field and $\overline{u_m}$ is its complex conjugate. For rotating wave states, $E_m$ are constant; and for a supercritical bifurcation, $E_m \propto Re - Re_c$ for $(Re - Re_c)/Re_c \ll 1$. Figure 2 shows how $E_m$ for the various branches of solutions in Table I vary with $Re$, clearly showing the onset of the three-dimensional states to be supercritical. Furthermore,
our numerical results for $H/R = 4.0$ are completely consistent with the experimental findings, raising questions about why the nonlinear simulations reported in Ref. 2 are in disagreement, given the similarities in spectral solution technique and resolution used in the present study. We do note that for these problems near the onset of three-dimensional flow, transients are very long, requiring temporal evolutions of multiple viscous times.

Beyond comparing onset data between our simulations and the experiments, we now compare the three-dimensional structure of the flow solutions with the PIV measurements from the experiments.\textsuperscript{4,6,7} The axisymmetric components of the three components of vorticity, $(\zeta_0, \eta_0, \zeta_0)$, of the states at $(Re,H/R) = (2300,3.5)$, (2900,4.6), and (3100,5.3) are shown in Figs. 3–5, respectively. Being so close to the bifurcation, these are very close to the corresponding vorticity of the basic state. These have features that are very much like those found in the basic states at lower $H/R$ (compare with vorticity structures in Ref. 22 for $H/R = 2.5$ where the basic state loses stability to axisymmetric time-periodic flow, which is subsequently unsteady to modulated rotating waves,\textsuperscript{23–25} and with Ref. 26 where at $H/R = 1.58$ the basic state is simultaneously unstable to $m = 0$ unsteady axisymmetric flow and $m = 2$ rotating wave states). The point is that the basic state features do not qualitatively change with $H/R$ (the flow is essentially stretched in the axial direction by a factor dependent on $H/R$), and they may have primary instability to either $m = 0$ or $m \neq 0$, or simultaneously to both. This was also borne out in the linear stability analysis of Ref. 8.

Instead of looking at the unstable basic state, we now consider the three-dimensional perturbations of the bifurcating states. These are obtained by subtracting the axisymmetric components (shown in Figs. 3–5) from the full three-dimensional solution. Very close to the onset of instability, these perturbation fields are very close to the bifurcation eigenfunctions. Columns (a) and (b) of

![FIG. 4. (Color online) Contours of the axisymmetric components of the vorticity for $Re = 2900$ and $H/R = 4.6$. There are 15 positive (solid/red) and 15 negative (dashed/yellow) contour levels in the range $[-1.0, 1.0]$ for each plot.](attachment:image.png)
Fig. 6 show the axial vorticity $\zeta$ and its three-dimensional perturbation $\zeta - \zeta_0$ at $z = 0.75H/R$ for RW3 and RW2 and at $z = 0.5H/R$ for RW4. Column (a) should be compared with the experimental PIV data taken at the same points in parameter space (see the top row of Fig. 8 in Ref. 7). So, one must conclude given the qualitative and quantitative agreement on critical $Re$, frequency and wavenumber as well as the spatial structure of the nonlinear three-dimensional states between our simulations and the experiments that both are describing the same nonlinear flows. Our analysis shows that these states are in fact rotating waves, of the same form as previously found by ourselves and others in shorter cylinders. A fuller view of their perturbation three-dimensional structure is presented in Fig. 7, showing isosurfaces of $\zeta - \zeta_0$ at $\pm 0.01$ for RW3, $\pm 0.005$ for RW2, and $\pm 0.002$ for RW2. We see that the perturbation axial vorticity is quite localized and comes in $m$ pairs of spiraled structures whose pitch angle varies considerably with axial distance, becoming more parallel to the axis with increased distance from the stationary top endwall. These rotate uniformly without change of shape (one complete turn in a period $2\pi m/\omega$, where $\omega$ is listed in Table I).

In the analysis of their experimental data, they assume that the enclosed swirling flow is made up of a combination of helical modes plus an axisymmetric “assigned” flow. It is not yet clear if these helical modes form a compete basis, even for swirling flows unbounded in the axial direction. Helical symmetry means that the flow is invariant to $H_{L,\lambda}$ whose action on the velocity is
where $2\pi l$ is the helix pitch. Helical symmetry requires an unbounded axial direction, and this is clearly not the case for the flows in the enclosed finite cylinder, and the question is whether this is even true locally. We have seen above that the three-dimensional flows that result from the instability of the steady axisymmetric basic state, both in the experiments and the simulations, are rotating waves, i.e., three-dimensional flows that are steady in an appropriate frame of reference. At the bifurcation, the continuous $SO(2)$ symmetry, $R_\theta$ and $U_s$, is broken and the bifurcating rotating wave solution has discrete symmetry $Z_m = R_{2\pi l}/2 \cdots R_{2\pi l/m}$, where $m$ is the azimuthal wavenumber of the solution. In time, the rotating wave is $2\pi/\omega$-periodic, but when viewed in an appropriate rotating frame of reference it is stationary, i.e., for a rotating wave of azimuthal wavenumber $m$,

$$
\mathbf{u}_{RWm}(r, \theta, z, t) = \mathbf{u}_{RWm}(r, \theta - 2\pi/m, z, t + 2\pi/\omega).
$$

They\(^4\) assume that the observed three-dimensional instabilities are due to the instability of a concentrated central vortex, their so-called assigned flow, which they assume has vorticity of the form,

$$
(\xi_\alpha, \eta_\alpha, \zeta_\alpha) = \left(0, \frac{r}{\Gamma_0}, \frac{\Gamma_0}{\pi R_0^2} \exp\left(-\frac{r^2}{R_0^2}\right), 0\right),
$$

where $\Gamma_0$, $R_0$, and $l$ are fitting parameters representing the vortex circulation, its core radius and a helix pitch. The flow depends only on $r$, it is axisymmetric, axial translation symmetric, and steady. Note also that the radial vorticity $\xi_\alpha$ is assumed to be zero, but in the present problem the radial
vorticity of the basic flow is non-trivial, and its strength is only a little weaker than the azimuthal or axial vorticity strength (see Figs. 3–5). They proceed to analyze their experimental PIV data by selecting some values for $C_0$, $R_0$, and $l$ (it is not clear what criteria is used to make this selection), and then at a specific axial location ($z = 0.75H/R$), they subtract $f_a$ from the PIV-measured $f$. The result shows at about mid-radius $m$ vortex structures all of the same sign which are interpreted as being helical vortices, multiplets. If we perform the same exercise on our rotating wave solutions (column (a) of Fig. 6), we obtain $f/C_0 f_a$ which are shown in column (c) of Fig. 6. These contours of $f/C_0 f_a$ are in excellent agreement with the corresponding experimental contours shown in the bottom row of Fig. 8 in Ref. 7. Similar experimental flows were earlier analyzed with similar results, but the fitting parameters used were not reported. We have tried a variety of values of fitting parameters, and the results are not too sensitive to their choice (within reason). What happens is that by subtracting the assigned flow $f_a$, which is a Gaussian distribution that approaches zero at about mid-radius, from the full $f$ amounts to giving the three-dimensional perturbation $f_p = f - f_0$ a strong bias. Now, $f_p$ consists primarily of alternating patches of positive and negative axial vorticity of comparable strengths (see column (b) of Fig. 6). By subtracting $f_a$, the resulting field has a reduced positive perturbation and an increased negative perturbation, and by a judicious selection of $R_0$ and $C_0$ one can locate where this bias is applied in order to produce what looks like multiplets (column (c) of Fig. 6). Note that in this exercise, the parameter $l$, which is related to the helix angle, plays no role.

In their interpretation of instability, they assume that the assigned flow has a constant helix angle, and further that the helix angles of the velocity $v/w$ and the vorticity $\eta/\zeta$ are the same. We note that the ratio of these two angles plays a very important role in vortex breakdown.27

IV. CONCLUSIONS

All of the states examined in the experiments are rotating waves that bifurcate supercritically at symmetry-breaking Hopf bifurcations from the steady axisymmetric basic state. They
have in fact verified that their experiments are consistent with this by detailed comparisons with linear stability analysis. Now, we have provided fully nonlinear numerical solutions of these bifurcating rotating waves and have shown that they agree very well with the experiments (using direct comparisons of the axial vorticity at the axial location where the PIV data was taken, the critical $Re$ for their onset, and their frequency as compared to the experimentally measured frequency using LDA). There is no doubt that these rotating waves in large $H/R$ are the same as those described in shorter cylinders with $H/R \sim 1.5$ and are related to the modulated rotating waves resulting from three-dimensional instabilities of the axisymmetric time-periodic states at intermediate $H/R$.\textsuperscript{1,9,24,25}

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