Coupling of the interfacial and bulk flow in knife-edge viscometers
Juan M. Lopez and Amir H. Hirsa

Citation: Physics of Fluids (1994-present) 27, 042102 (2015); doi: 10.1063/1.4916619
View online: http://dx.doi.org/10.1063/1.4916619
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/27/4?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
Gravity-driven film flow with variable physical properties
Phys. Fluids 18, 083602 (2006); 10.1063/1.2241950

Shear-driven and pressure-driven flow of a nematic liquid crystal in a slowly varying channel
Phys. Fluids 18, 027105 (2006); 10.1063/1.2145750

Dynamics and stability of a thin liquid film on a heated rotating disk film with variable viscosity
Phys. Fluids 17, 102103 (2005); 10.1063/1.2099007

The flow of thin liquid films over spinning disks: Hydrodynamics and mass transfer
Phys. Fluids 17, 052102 (2005); 10.1063/1.1891814

Hydraulic jumps and standing waves in gravity-driven flows of viscous liquids in wavy open channels
Phys. Fluids 16, 3868 (2004); 10.1063/1.1789431
Coupling of the interfacial and bulk flow in knife-edge viscometers

Juan M. Lopez\textsuperscript{1,}\textsuperscript{a} and Amir H. Hirsa\textsuperscript{2}

\textsuperscript{1}School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287, USA
\textsuperscript{2}Department of Mechanical, Aerospace and Nuclear Engineering, Rensselaer Polytechnic Institute, Troy, New York 12180-3590, USA

(Received 10 June 2014; accepted 17 March 2015; published online 3 April 2015)

The operation of the knife-edge viscometer requires knowledge of the interfacial velocity profile in order to determine the viscous traction between the surface film and the knife edge and hence measure the surface shear viscosity of the film. The interfacial velocity profile can be obtained analytically in two limiting regimes. One is the limit of the surface shear viscosity going to infinity, in which case the interfacial velocity profile is independent of the bulk flow and a simple analytic expression is available. The other limit corresponds to vanishing bulk flow inertia, allowing one to reduce the Navier–Stokes equations to the Stokes equation, and the resulting linear system can be solved analytically. For finite inertia and finite surface shear viscosity, the knife-edge viscometer hydrodynamics is governed by the coupled nonlinear set of equations. Here, we study these numerically, explore the coupling between the interfacial and bulk flow, and delineate the ranges of surface shear viscosity and knife-edge rotation rates where the analytic approximations are appropriate. We also examine a variant of the knife-edge viscometer, known as the double-wall ring viscometer, which is essentially the same geometry but with the addition of a stationary inner cylinder so that the bulk fluid is contained in an annular channel rather than a cylinder. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4916619]

I. INTRODUCTION

After more than 50 yr of investigating how to measure surface (excess) shear viscosity, it remains a controversial issue. The complications stem from the fact that to measure a viscous response the system needs to be flowing, and for a surface film on a liquid substrate, this means that the liquid in the bulk will also be flowing. It is widely recognized that the interfacial flow is not independent of the flow in the bulk. In other words, the bulk liquid is not merely supporting the interfacial film, and that for quantitative measurement of surface shear viscosity, the coupling between interfacial and bulk flow must be accounted for properly.

The knife-edge viscometer is one of several devices used to measure the surface shear viscosity of interfacial films.\textsuperscript{1,2} As is true of most viscometers, the device relies on a direct measurement of some quantity (velocity or torque) which is compared to a theoretical value determined for a range of surface viscosities and by matching the two, an estimate of the viscosity is made. For the most part, the theoretical value can only be obtained analytically when the governing equations are linear, which for hydrodynamic systems, this means that the flow is inertialless. Another very useful surface viscometer is the deep-channel viscometer.\textsuperscript{1–3} Its use had also for the most part been restricted to operation within the inertialless flow limit, until a numerical solution for the nonlinear flow problem was presented, significantly extending its range of operation and signal-to-noise ratio.\textsuperscript{4–6} The deep-channel viscometer consists of an annular region filled with a bulk liquid (typically water), the upper surface of which

\textsuperscript{a}Electronic address: jmlopez@asu.edu
has a surface film. The flow is driven by the constant rotation of the annulus floor whilst the annular walls are stationary. The intrinsic surface shear viscosity is related to the angle at which the vortex lines emanating from the rotating floor meet the monolayer. For an inviscid monolayer, the vortex lines are normal to it. For very viscous monolayers, the vortex line tends to become tangential to the interface and the entire monolayer is stationary, said to be frozen. This can present a practical limit in using a deep-channel viscometer.

The knife-edge viscometer comes in a number of variants, but they all basically rely on the viscous traction between a sharp edge and the surface film that it just impinges upon. The variation basically amounts to whether the sharp edge is stationary and the cylinder is rotating, or if the sharp edge is rotating and the cylinder is stationary. No matter how viscous the film is, there is differential motion between it and the sharp edge, and so there will be a viscous response. These set-ups avoid the “frozen” surface film limitations of the deep-channel viscometer. In the inertialess flow limit, there is no essential difference between the rotating knife edge or the rotating cylinder set-ups, as the equation governing the azimuthal velocity is the same and is decoupled from the meridional flow (axial and radial velocities) equations. Once the azimuthal flow is determined, the meridional flow could then be determined, but most implementations of the knife-edge viscometer do not do this. Instead, they generally set the meridional bulk flow to zero. This is ostensibly because the coupled nonlinear system is too difficult to solve analytically.\(^7,8\) This coupling is present and non-negligible even in the limit of inertialess flow; the vortex lines in the system due either to the rotating knife edge or rotating cylinder will experience gradients in the axial \(z\)-direction (vortex line bending) in some parts of the flow, resulting in a meridional flow. In this paper, we explore this coupling numerically, giving a comprehensive overview of the knife-edge viscometer flow.

Most surface shear viscometers operate in the large Boussinesq number limit (ratio of surface shear viscosity to bulk viscosity times an appropriate length scale), where it is anticipated (hoped) that the coupling between the interfacial and bulk flow is negligible.\(^9,10\) It is widely accepted that operation with \(Bo \sim 1\) requires a careful treatment of the full hydrodynamic problem, and this has not been done for many viscometers.\(^11–13\)

Even when the coupling is attempted for operations at finite Reynolds numbers, the governing equations for the bulk hydrodynamics used are the Navier–Stokes equations neglecting the nonlinear advection terms.\(^11,14–16\) Other attempts to account for the coupling assume the bulk flow under idealized conditions rather than calculating the coupled problem in the actual geometry, and this may lead to errors in interpreting measurement results.\(^17–19\)

II. GOVERNING EQUATIONS

Consider the flow in a stationary cylinder, of radius \(R\), filled with water of dynamic viscosity \(\mu\) and kinematic viscosity \(\nu\) to a depth \(H\). The top surface is uniformly covered by a thin film and the flow is driven by the constant rotation at \(\Omega \text{ rad/s}\) of a knife edge of radius \(a\) and edge thickness \(\epsilon a\) that just touches the film (see Fig. 1 for a schematic). The governing equations are the Navier–Stokes equations with no-slip boundary conditions on the cylinder bottom and sidewall, together with the tangential stress balance at the air-water interface, and no-slip at the rotating knife.

Using \(a\) as the length scale and \(1/\Omega\) as the time scale, and assuming the flow to remain axisymmetric, the non-dimensional Navier–Stokes equations are written in cylindrical coordinates \((r, \theta, z)\), with the velocity in terms of the stream-function \(\psi\),

\[
\mathbf{u} = (u, v, w) = (-\partial\psi/\partial z, \gamma, \partial\psi/\partial r)/r, \tag{1}
\]

and the vorticity

\[
\nabla \times \mathbf{u} = (-\partial\gamma/\partial z, r\eta, \partial\gamma/\partial r)/r, \tag{2}
\]

where \(\gamma = rv\) is the angular momentum and \(\eta\) is the azimuthal vorticity, resulting in

\[
\frac{\partial \gamma}{\partial t} - \frac{1}{r} \frac{\partial}{\partial z} \left( \frac{\partial \gamma}{\partial r} \right) + \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial \gamma}{\partial z} = \frac{1}{Re} \left( \frac{\partial^2 \gamma}{\partial z^2} + \frac{\partial^2 \gamma}{\partial r^2} \right) - \frac{1}{r} \frac{\partial \gamma}{\partial r}, \tag{3}
\]
The air-water interface is described using the Boussinesq–Scriven surface model,\textsuperscript{1,2,20} which states that the surface stress tensor is
\[
\hat{T}^s = (\sigma + (\kappa^s - \mu^s) \text{div}_s \mathbf{u}^s) \mathbf{I}_s + \mu^s (\nabla_s \mathbf{u}^s \cdot \mathbf{I}_s + \mathbf{I}_s \cdot (\nabla_s \mathbf{u}^s)^T),
\]
where $\sigma$ is the equilibrium surface tension, $\mu^s$ is the surface shear viscosity and $\kappa^s$ is the surface dilatational viscosity, $\mathbf{u}^s$ is the surface velocity vector, $\text{div}_s$ is the surface divergence operator, $\nabla_s$ is the surface gradient operator, and $\mathbf{I}$ projects any vector onto the interface. For a flat interface, only the tangential stress balance plays a dynamic role. The non-dimensional tangential stress balance, at the interface $z = A_z$, in the azimuthal direction is
\[
\frac{\partial v}{\partial z} = \mu^s \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + \frac{\partial^2 \hat{v}}{\partial r^2} \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right),
\]
where $\Omega = a^2/\nu$ is the Reynolds number giving the ratio of viscous time, $a^2/\nu$, to inertial time, $1/\Omega$. Geometric parameters describing this system are the ratio of cylinder radius to outer knife-edge radius, $A_r = R/a$, and water depth to outer knife-edge radius, $A_z = H/a$. These are taken to be $A_r = A_z = 2$ as the results are not very sensitive to these. Another geometric parameter is the ratio of the knife-edge thickness to its outer radius, $\epsilon$, and this is typically vanishingly small. We use a finite-difference discretization of the governing equations (detailed below), and define the knife-edge thickness as $\epsilon = A_r (n_{\text{edge}} - 1)/(n_r - 1)$, where $n_r$ is the number of grid points used in the radial direction and $n_{\text{edge}}$ is the number of grid points covering the width of the knife edge that is in contact with the interface. So, a sharp knife edge which touches the interface at a point has $\epsilon = 0$. For most of the results presented here, we have used $n_r = 101, n_{\text{edge}} = 1$, and $A_r = 2$, corresponding to $\epsilon = 0$. The effects of finite epsilon are described at the end of Sec. III.

The no-slip boundary conditions on the stationary cylinder side and bottom are
\[
\text{Side, } r = A_r : \quad \gamma = \psi = 0, \quad \eta = -\frac{1}{A_r} \frac{\partial^2 \psi}{\partial r^2},
\]
\[
\text{Bottom, } z = 0 : \quad \gamma = \psi = 0, \quad \eta = -\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2}.
\]
and in the radial direction

\[
\eta = \frac{1}{Ca} \frac{\partial \sigma}{\partial r} + (\hat{\mu}^s + \hat{k}^s)(\frac{1}{r^2} \frac{\partial^2 \psi}{\partial r \partial z} - \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial}{\partial z} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \frac{\partial (\hat{\mu}^s + \hat{k}^s)}{\partial r} + \frac{2}{r^2} \frac{\partial \psi}{\partial r} \frac{\partial \hat{\mu}^s}{\partial r}),
\]

where \( Ca = \mu \Omega a / \sigma_0 \) is the capillary number, \( \sigma = \sigma / \sigma_0 \), \( \sigma_0 \) is the surface tension of the interface in the absence of a film, \( \hat{\mu} = \mu / \mu_a \) and \( \hat{k} = k / \mu a \). We have previously demonstrated through experiments and computations\(^4\)\(^6\) that at steady state, only a minuscule amount of Marangoni stress is required in order for the surface tension gradient to eliminate any radial component of surface velocity. This is due to the small capillary number in the flows of interest.\(^6\) This was shown in a deep-channel viscometer geometry consisting of an annular region bounded by stationary cylinders and a rotating floor. The secondary meridional flow is much weaker in the present knife-edge geometry than in the deep-channel; therefore at steady state, the radial component of velocity is expected to be zero throughout the interface. Under such conditions, (10) is replaced\(^6\) by

\[
\eta = \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2},
\]

any concentration gradients of the surface film are negligible, and the surface dilatational viscosity no longer appears in the governing equations. Then, \( \hat{\mu}^s \) is often termed the Boussinesq number \( Bo = \mu^s / \mu a \), and the azimuthal tangential stress equation at the interface reduces to

\[
\frac{\partial v}{\partial z} = Bo \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right).
\]

The Navier–Stokes equations, Eqs. (3)–(5) are solved in the bulk using a second-order centered finite-difference discretization in space and a second-order predictor-corrector (Heun’s method) for time advancement. The method has been used and tested on internal rotating flows.\(^{2,3}\) It has been extended to handle viscous and inviscid surface films.\(^4\)\(^6\) The same implementation is used here, where Eqs. (11) and (12) are imposed at the interface. Knowing the interior bulk flow at any point in time, Eq. (11) is evaluated using second-order one-sided differences. As in other implementations,\(^6\) Eq. (12) is solved for \( v_s(r,z) = v(r,z = 0) \) at each point in time with the just-computed interior solution \( v(r,z) \) which is used to determine \( \partial v / \partial z \mid_{z = 0} \) using second-order one-sided differences. The difference in the present implementation is that the interface is not a singly connected surface due to the impinging knife edge. This means that Eq. (12) needs to be solved for \( r \in (0, 1 - \epsilon) \) with boundary conditions \( v_s(0) = 0 \) and \( v_s(1 - \epsilon) = 1 - \epsilon \), and then for \( r \in (1, A_r) \) with \( v_s(1) = 1 \) and \( v_s(A_r) = 0 \). At the knife edge, \( v_s = r \) for \( r \in (1 - \epsilon, 1) \). In the limit as \( Bo \to \infty \), the interfacial hydrodynamics decouples from the bulk hydrodynamics (but the bulk remains coupled to the interface), and Eq. (12) reduces to

\[
\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} = 0,
\]

which has a simple analytic solution of the form \( v_s(r) = \alpha r \beta/r \), where \( \alpha \) and \( \beta \) are constants that depend only on the geometric parameters \( A_r \) and \( \epsilon \); for \( \epsilon = 0 \), \( \alpha = 1 \), and \( \beta = 0 \) for \( r \in (0, 1) \), and \( \alpha = 1 - A_r^2 / (A_r^2 - 1) \) and \( \beta = A_r^2 / (A_r^2 - 1) \) for \( r \in (1, A_r) \). In the limit \( A_r \to \infty \), \( \alpha \to 0 \), and \( \beta \to 1 \), leading to Rankin vortex flow at the interface in the double limit \( Bo \to \infty \) and \( A_r \to \infty \). For the finite \( A_r = 2 \), we shall focus on here, \( \alpha = 1 \) and \( \beta = 0 \) for \( r \in (0, 1) \) and \( \alpha = -1/3 \) and \( \beta = 4/3 \) for \( r \in (1, 2) \).

In the absence of any surface film, the air-water interface can be idealized as a stress-free interface, where \( \eta = 0 \) and \( \partial v / \partial z = 0 \). This case is distinguished from that of an inviscid surface film \( (Bo = 0) \) by having non-zero surface radial velocity. For the inviscid film, elasticity forces the surface radial velocity to be zero.

Most of the results presented here have been computed with \( n_r = n_z = 101 \) grid points in the \( r \) and \( z \) directions. We have verified that doubling the grid resolution does not lead to appreciable differences in the solutions. The time step \( \delta t \) used depends on \( Re \) (and the spatial resolution); for \( Re < 10^2 \delta t \approx 1/Re \) and for \( Re > 10^2 \), we have kept \( \delta t = 5 \times 10^{-3} \). The value used for \( \delta t \) is primarily dictated by numerical stability requirements, as the second-order method is explicit. Steady state is typically reached, when starting from rest, in about one viscous time (\( \alpha^2 / \nu \) seconds, or in non-dimensional time units, \( t = Re \)).
III. RESULTS

At low $Re = 10$, comparing the stress-free case with the very low surface shear viscosity $Bo = 10^{-4}$ case, Fig. 2 shows that the angular momentum, $\gamma$, diffuses into the bulk from the knife edge through the action of viscosity in the bulk; the vortex lines (contours of $\gamma$) associated with these two cases are virtually indistinguishable. In both cases, they meet the interface at right angles. However, the secondary flows that are driven by the axial $z$-gradients in $\gamma$, i.e., driven by vortex line bending, are different between the two. This is due to the azimuthal vorticity, $\eta$, boundary conditions on the
FIG. 3. Radial profiles of the azimuthal velocity at the interface $v_s$ and the interfacial shear $S$ for $Bo$ and $Re$ as indicated, all with $\epsilon = 0$. In part (e), the analytic solution to Eq. (13) obtained in the limit $Bo \to \infty$ is also included.

interface. The stress-free interface has $\eta = 0$ whereas for the cases with films, $\eta$ is given by Eq. (11). The secondary meridional flow in both cases consists of a radial outflow near the interface driven by the rotation of the knife edge, and results in a large scale circulation cell throughout the entire domain, as illustrated by the streamlines, $\psi$. Note that the meridional flow in the stress-free case is stronger than the low $Bo$ case, and this is due to the radial surface velocity not being restricted to zero in the stress-free case.

Increasing the surface viscosity to $Bo = 1$ results in the vortex lines no longer meeting the interface at right angles, and these axial $z$-gradients in $\gamma$ at the interface result in enhanced azimuthal vorticity across the interface. As a consequence, the meridional overturning cell is significantly stronger; see Fig. 2. At a larger surface shear viscosity $Bo = 100$, the film for $r < 1$ rotates as a solid body at the same rate as the knife and the vortex lines emanating from the rotating film enter deep into the bulk and since they cannot terminate on the stationary cylinder walls or in the interior, they emerge
at the interface with $r > 1$, causing the outer film to rotate with strong shear. This vortex line bending in the interior generates azimuthal vorticity $\eta$ (via the $2\gamma/r^3\partial\gamma/\partial z$ term in Eq. (4)) and this vorticity drives the overturning meridional circulation, $\psi$ (via Eq. (5)). For small $Re$, this secondary meridional flow is weak (its strength scales with $Re$) and it does not affect the vortex lines appreciably.

Figure 3 details the interfacial flows at various $Re$ and $Bo$ for $\epsilon = 0$. Specifically, the left panels of the figure show the azimuthal velocity (the only non-zero component of velocity) at the interface, $v_\phi = v(r, z = A_\epsilon)$, at steady state, and the right panels show the corresponding interfacial shear, $S = \partial v_\phi/\partial r - v_\phi/r$. For the knife edge of thickness $\epsilon = 0$, the azimuthal velocity along the surface of the knife edge is $v_\phi(1) = 1$. For very small $Bo = 10^{-4}$, there is some shear across the entire interface,
FIG. 5. Contours of $\gamma$, $\eta$, and $\psi$ for $Re = 1000$, $\epsilon = 0$ with a stress-free interface as well as cases with $Bo$ as indicated. There are 20 positive (red/lighter) and 20 negative (blue/darker) contour levels in the ranges $\gamma \in [0, 1]$, $\eta = \pm 15$, and $\psi = \pm 4 \times 10^{-2}$.

but it increases rapidly near the knife edge. As expected, the shear is of opposite signs either side of the knife edge and $v_s$ is not symmetric about the knife edge. There are two contributions to this asymmetry, one is due to the curvature associated with the cylindrical geometry and the second is due to the secondary meridional flow in the bulk. At the low $Re = 10$, the second contribution is small, but it dominates when $Re$ is larger. The shear rapidly drops to zero towards $r = 0$, indicating that the film near the axis is rotating as a solid body (but at a rate considerably slower than the knife). The region of near solid-body rotation of the film extends from the axis to the knife edge with increasing $Bo$, so that for viscous films, the knife edge rotation makes the film on the inner region spin like a solid body at the same rate as the knife edge. For $Bo = 100$, the computed $v_s$ and $S$ for all $Re$ is virtually
indistinguishable from the analytic solution obtained in the limit of $Bo \to \infty$. In contrast, the surface at radii larger than the knife edge (the outer region) is subjected to significant shear. The difference in behavior between the inner and outer regions of the interface is due to the film being pinned to the stationary cylinder while being driven into motion by the rotating knife edge, resulting in the large interfacial shear, whereas at the axis, the film is free to rotate and hence the knife edge drives the whole inner region as a solid body when the film is sufficiently viscous.

Prior to the comparison between nonlinear results and the analytic profile obtained in the limit $Bo \to \infty$, shown in Fig. 3, one did not have a concrete idea about how well the analytic theory captured behavior at finite $Re$ and $Bo$. These parameters are always finite in any physical experiment. In the $Bo \to \infty$ limit, the interface is decoupled from the bulk, however for large but finite $Bo$, the bulk and interfacial flows remain coupled, and the question then arises as to how good an approximation is the analytic profile for a given large but finite $Bo$ at different $Re$. Figure 3 illustrates how the approximation deteriorates with increasing $Re$ at various $Bo$.

Increasing $Re$ to 100, Fig. 4 shows that the meridional flow generated by the vortex line bending that at low $Re$ resulted from the stationary cylinder boundary conditions, now is strong enough to advect the vortex lines with it, resulting in further vortex line bending particularly near the interface. This is further manifested in the angles at which the vortex lines meet the interface (compare the $\gamma$ contour for $Bo = 1$ in Figs. 2 and 4). This difference is due to the different axial gradients in the azimuthal velocity at the surface due to the different radial gradients in the azimuthal velocity driven by the stronger bulk meridional flow at the larger $Re$, even though both cases have the same $Bo$ (see Eq. (12)).

For $Re = 1000$, the secondary meridional flow becomes dominant. Figure 5 shows that the fluid in the bulk for all $Bo$, as well as for the stress-free interface case, is being centrifuged radially outward by the much faster rotation of the knife edge. The fluid jetted by the knife edge meets the outer stationary cylinder wall and is deflected deep into the bulk, setting up the secondary meridional flow; see the streamlines (contours of $\psi$) in Fig. 5.

The bulk flow in all of the above discussed results is purely due to the rotating knife edge. As a measure of the strength of the meridional bulk flow, we use the largest positive value of the azimuthal vorticity, $\eta_{\text{max}}$. Figure 6(a) shows how $\eta_{\text{max}}$ varies with $Re$ and $Bo$ over several decades, and Fig. 6(b) details how $\eta_{\text{max}}$ varies with $Re$ for a given $Bo = 1$. As expected, for very low $Re$, the induced secondary flow in the bulk varies linearly with $Re$, up to $Re \approx 20$; see Fig. 6(b). At this $Re$, the induced meridional flow is of magnitude order one (i.e., $\eta_{\text{max}} \approx 1$), and is no longer negligible. From Fig. 6(a), we see that the secondary bulk flow is much greater for large $Bo$ than for low $Bo$, and this effect is stronger for low $Re$, and the transition occurs smoothly for $Bo$ between $10^{-2}$ and 1, while for larger or lower $Bo$, the strength of the induced secondary bulk flow becomes independent of $Bo$. The reason for the stronger meridional flows at large $Bo$ is that for the very viscous films, the interface encircled by the knife edge spins like a solid body, $v_s(r) = r$, and the interfacial flow is equivalent to that corresponding to a solid disk of the same radius as the knife edge rotating at the air-water interface.

![FIG. 6. Variation of the maximum (positive) azimuthal vorticity, $\eta_{\text{max}}$, (a) with $Bo$ for $Re$ as indicated and (b) with $Re$ for $Bo = 1$, all with $\epsilon = 0$.](image)
limit of the knife edge radius being equal to the cylinder radius ($A_r \to 1$), for large $Bo$, we recover the flow in an enclosed stationary cylinder driven by a rotating endwall.\(^{21}\) For larger $A_r$, the flow is similar to a partially rotating endwall,\(^{22}\) except that in that case, the azimuthal velocity beyond the edge of the rotating disk endwall is zero, whereas for our very viscous film, it is not zero and is given by $v_s = \alpha r + \beta/r$, as described in Sec. II.

So far, all of the results discussed correspond to zero knife-edge thickness. Now we consider the effects of finite knife-edge thickness. Figure 7 shows the interfacial velocity and shear profiles for $Re = 100$ and $Bo = 1$ and various thicknesses $\epsilon$. The outer profiles (for $r > 1$) are virtually independent of $\epsilon$. The inner $v_s$ profiles, however, do show variation with $\epsilon$, and this is because $Bo = 1$ is not large enough to produce an inner film moving as a solid disk. This aspect is more readily appreciated from Fig. 8, which shows $v_s(r)$ and $S(r)$ with fixed $Re = 100$ and $\epsilon = 0.08$ for various $Bo$. For $Bo = 100$, $v_s(r) = r$ for $r < 1$, showing that the inner film rotates as a solid disk, and this is true for any $\epsilon$. For less viscous films, this is no longer true. These results are consistent with experimental measurements of torque in a knife edge surface viscometer, in that the measurements are relatively insensitive to the width of a finely ground knife edge.\(^{23}\)

### IV. DOUBLE-WALL RING (DWR) VISCOMETER

A variant of the knife edge viscometer is the so-called DWR viscometer, which is essentially the same apparatus as shown schematically in Fig. 1, but with the addition of a stationary inner cylinder so that the bulk fluid is contained in an annular channel rather than a cylinder. The addition of the inner cylinder enhances the sensitivity of the viscometer.

Let the non-dimensional radius of the inner cylinder be $R_i$, a fraction less than one of the radius of the knife edge. The governing equations for the two cases are the same, but they differ in their boundary conditions at their respective inner radius. For the knife edge without an inner cylinder, we have $\psi = 0$, $\eta = 0$, and $v = 0$ at $r = 0$ (axis condition for axisymmetric flow), whereas for DWR, we have $\psi = 0$, $\eta = -1/R_i \partial^2 \psi/\partial r^2$, and $v = 0$ at $r = R_i$ (no-slip at the stationary inner cylinder). The numerical
solution technique is the same as for the knife edge without an inner cylinder (using second-order one-sided finite difference approximation for the boundary condition on $\eta$ at the inner cylinder).

Figure 9 shows results for DWR with $A_r = 2$, $Re = 100$, and $Bo = 1$ (the same as for the knife edge without an inner cylinder shown in the $Bo = 1$ row of Fig. 4), for various values of $R_i$. This choice of $Re$ and $Bo$ is in the mid-range of those examined in Sec. III. The interfacial flow for $r \geq 1$ is the same for both the knife edge without an inner cylinder and DWR, but they differ for $r < 1 - \epsilon$. For the knife edge without an inner cylinder, in the limit of $Bo \to \infty$, $v_s(r) = r$ and the surface shear $S = 0$; the highly viscous surface film rotates as a solid body with the knife edge. In contrast, for DWR, such solid-body rotation is not possible as the interface is brought to rest at the inner cylinder; $v_s(r) = \alpha r + \beta/r$ and $S(r) = -2\beta/r^2$ with $\alpha = 1 - R_i^2/(R_i^2 - 1)$ and $\beta = R_i^2/(R_i^2 - 1)$, for $r \in [R_i, 1 - \epsilon)$. In the limit $R_i \to 0$, the surface velocity and shear in the two cases are the same, but their bulk flow differs due to the viscous boundary layer on the stationary inner cylinder (which would be an infinitely thin rod). Making the radius of the inner cylinder larger simply slows down the meridional flow in the interior via the viscous boundary layer on the inner cylinder, as is to be expected.

Comparing the azimuthal surface velocity profiles, $v_s$, and the corresponding interfacial shear $S$ between the DWR (Fig. 10) and the knife edge without an inner cylinder (Fig. 3), it is clear that there is no essential difference in the sensitivity of $v_s$ and $S$ to variations in $Bo$ between the two cases.
Having more interfacial shear inside the knife edge \((r < 1 - \epsilon)\) in DWR does not improve the flow sensitivity to variations in \(Bo\). As far as operation as a surface viscometer, there is no significant difference between DWR and the knife edge without an inner cylinder, and the smaller the container is relative to the radius of the knife edge, the more influence there is from the cylinder boundary layers.

V. CONCLUSIONS

The coupling between the interfacial flow of a thin viscous film, driven by the rotation of a circular knife edge, and the underlying bulk liquid has been explored numerically, retaining the nonlinear terms in the Navier–Stokes equations. The interfacial flow is described in terms of a Boussinesq–Scriven surface model. This is in contrast to past analyses of the knife-edge viscometer in which only the azimuthal component of the velocity is considered to be non-zero. This means that under those conditions, the only non-zero component of vorticity is vertical (in the axial direction), and the only mechanism considered in setting the film in motion is the viscous traction between the film and the knife edge. This is a reasonable approximation for exceedingly viscous films, but for finitely viscous films, there is an additional mechanism at play. The vortex lines which emanate from the knife edge into the bulk cannot terminate in the interior of the bulk flow nor can they terminate on the stationary cylinder walls. They must terminate at the air-water interface, and for this to happen, they have to bend. So, the vorticity must have non-zero components in the radial and azimuthal directions. These then drive a secondary meridional flow, i.e., the radial and axial components of velocity are not zero. The question then becomes how important is this secondary meridional flow. As described above, it is not important when the film is very viscous. Our study indicates that when the surface film’s shear viscosity is about 100 times larger than the bulk fluid’s dynamic viscosity multiplied by the radius of the rotating knife edge, then the interfacial flow is unaffected by the bulk flow, no matter how fast the knife is rotating. Furthermore, for less viscous films, the vortex line bending induced secondary flow is negligible if the knife is rotating slowly enough, so that the bulk flow is in the inertial-less regime. For water at room temperature in a centimeter scale knife-edge viscometer, the rotation rate of the knife has to be less than \(10^{-2}\) rad/s. This corresponds to the knife rotating at about one tenth the rate of the minute hand of a clock.

From a practical point of view, the knife edge has a finite thickness and the results show little sensitivity to finite thicknesses, in accord with prior analysis. Likewise, the addition of an inner cylinder has no significant effect on the coupling between the bulk and interfacial flow. In either configuration, in order to determine the interfacial velocity and shear, the coupling with the bulk flow, including the secondary meridional flow, must be properly accounted for. Furthermore, to achieve sensitivity in determining the surface shear viscosity of a given film, the radius of the knife edge has to be selected such that \(Bo\) is of order 0.1, with sensitivity dropping off for larger and smaller \(Bo\).

In measuring the interfacial flow, it is desirable to have as large a signal-to-noise ratio as possible,
and this is achieved by increasing \( Re \). The limitation on how large \( Re \) can be stems from the fact that the secondary meridional bulk flow becomes dynamically important with increasing \( Re \), and for \( Re \) much larger than order \( 10^3 \), the flow becomes time-dependent and three-dimensional, limiting its utility as a viscometer.

**ACKNOWLEDGMENTS**

This work was supported by the National Science Foundation Grant Nos. CBET-1064644 and CBET-1064498 and NASA Grant No. NNX13AQ22G. We thank Aditya Raghunandan for helpful comments on the manuscript.